Two Applications of Bisets and Idempotents for Fusion Systems

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Fusion Days
Amiens 2009

- S_1, S_2 groups
- (S_1, S_2)-biset = a set X with (free) compatible right S_1-action and left S_2-action
- Transitive (S_1, S_2)-bisets: for $P \leq S_1$, $\varphi\colon P \to S_2$ injective

$$S_2 \times_{(P,\varphi)} S_1 = S_2 \times S_1 / \sim, \quad (y\varphi(u), x) \sim (y, ux) \quad \text{for } x \in S_1, y \in S_2, u \in P.$$

- An (S_1, S_2)-bisets X can be viewed as a (left) $(S_1 \times S_2)$-set via $(u, v) \cdot x = vxu^{-1}$.
- Transitive $(S_1 \times S_2)$-sets: for $P \leq S_1$, $\varphi\colon P \to S_2$ injective

$$S_1 \times S_2 / \Delta_P^{\varphi}, \quad \Delta_P^{\varphi} = \{(x, \varphi(x)) \mid x \in P\}$$

- $[X]$ = the isomorphism class of (S_1, S_2)-bisets containing X
- $[P, \varphi]_{S_1}^{S_2} = [S_2 \times_{(P,\varphi)} S_1]$

- $A(S_1, S_2)$ = the Grothendieck group of isomorphism classes of (S_1, S_2)-bisets = the free abelian group on the basis elements $[P, \varphi]_{S_1}^{S_2}$
- $\circ\colon A(S_2, S_3) \times A(S_1, S_2) \to A(S_1, S_3), \quad [Y] \circ [X] = [Y \times_{S_2} X]$

$$[Q, \psi]_{S_2}^{S_3} \circ [P, \varphi]_{S_1}^{S_2} = \sum_{x \in [Q \backslash S_2 / \varphi(P)]} [\varphi^{-1}(\varphi(P) \cap Q^x), \psi \circ c_x \circ \varphi]_{S_1}^{S_3}$$

- $\epsilon\colon A(S_1, S_2) \to \mathbb{Z}, \quad \epsilon([X]) = |S_2 \backslash X| = |X|/|S_2|$

$$\epsilon([Y] \circ [X]) = \epsilon([Y])\epsilon([X])$$

- $A(S, S)$ = *double Burnside ring* of the group S (wrt the product $\circ$)
- $A^+(S, S)$ = the set of the elements of $A(S, S)$ with nonnegative coefficients (wrt the basis elements $[P, \varphi]_S^S$)
- $A(S, S)_{(p)} = A(S, S) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

- $\mathcal{F}$ fusion system on a finite p-group S
- $A_{\mathcal{F}}(S,S) = \mathbb{Z}$-span of $\{[P,\varphi]_S^S \mid P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\}$
- $A_{\mathcal{F}}(S,S)$ is a subring of $A(S,S)$.
- An element $\omega \in A(S,S)$ is called *right $\mathcal{F}$-stable* if

$$\omega \circ [P,\varphi]_P^S = \omega \circ [P,\iota_P]_P^S \quad \text{whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S);$$

similarly $\omega \in A(S,S)$ is called *left $\mathcal{F}$-stable* if

$$[\varphi(P),\varphi^{-1}]_S^P \circ \omega = [P,\mathrm{id}_P]_S^P \circ \omega \quad \text{whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S).$$

If $\omega = [X] \in A^+(S,S)$, then ω is right $\mathcal{F}$-stable if

$$X|_{(S,\varphi)} \cong X|_{(S,P)} \text{ as } (S,P)\text{-bisets whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S);$$

ω is left $\mathcal{F}$-stable if

$$X|_{(\varphi^{-1},S)} \cong X|_{(P,S)} \text{ as } (P,S)\text{-bisets whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S).$$

Theorem (Broto, Levi, Oliver)

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then there are elements $\omega \in A^{+}_{\mathcal{F}}(S,S)$ with $\epsilon(\omega) \not\equiv 0 \mod p$ which are both left and right $\mathcal{F}$-stable.

We call such an ω a characteristic biset for $\mathcal{F}$.

Example

Let $\mathcal{F} = \mathcal{F}_S(G), S \in \mathrm{Syl}_p(G)$. Then

$$\omega = [G] = \sum_{x \in [S \backslash G / S]} [S \cap S^x, c_x]_S^S$$

Theorem (Ragnarsson)

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then

1. *There is a unique idempotent $\omega_{\mathcal{F}} \in A_{\mathcal{F}}(S,S)_{(p)}$ with $\epsilon(\omega_{\mathcal{F}}) = 1$ that is both left and right $\mathcal{F}$-stable.*
2. *An idempotent $\omega \in A_{\mathcal{F}}(S,S)_{(p)}$ with $\epsilon(\omega) = 1$ is left $\mathcal{F}$-stable iff it is right $\mathcal{F}$-stable.*

We call $\omega_{\mathcal{F}}$ the *characteristic idempotent* of $\mathcal{F}$.

Definition

Let $\mathcal{F}$ be a fusion system on a finite p-group S.

1. $A^p_{\mathcal{F}}(S) = [S, \mathcal{F}] := \langle [Q, \mathrm{Aut}_{\mathcal{F}}(Q)] \mid Q \leq S \rangle$ the $\mathcal{F}$-*focal subgroup* of S
2. $O^p_{\mathcal{F}}(S) := \langle [Q, O^p(\mathrm{Aut}_{\mathcal{F}}(Q))] \mid Q \leq S \rangle$ the $\mathcal{F}$-*hyperfocal subgroup* of S

Theorem (Yoshida; Isaacs; Díaz, Glesser, P, Stancu)

Let S be a finite p-group which does not have $C_p \wr C_p$ as a homomorphic image. For every fusion system $\mathcal{F}$ on S,

$$[S, \mathcal{F}] = [S, N_{\mathcal{F}}(S)].$$

Theorem (Tate; Gagola, Isaacs; Díaz, Glesser, P, Stancu)

Let $\mathcal{F}$ be a fusion system on a finite p-group S, and let $\mathcal{H}$ be a subsystem of $\mathcal{F}$ on S. Then

$$[S, \mathcal{F}] = [S, \mathcal{H}] \iff O^p_{\mathcal{F}}(S) = O^p_{\mathcal{H}}(S).$$

In particular, $\mathcal{F} = \mathcal{F}_S(S)$ *iff* $[S, \mathcal{F}] = [S, S]$.

Transfer Maps

- For G, H groups, A abelian p-group, and

$$\omega = \sum_{i \in I} c_i [P_i, \varphi_i] \in A(G,H)_{(p)},$$

define

$$H^*(\omega) = \sum_{i \in I} c_i \mathrm{tr}_{P_i}^{G} \circ \varphi_i^* : H^*(H,A) \to H^*(G,A)$$

- If K is another group and $\eta \in A(H,K)_{(p)}$, we have

$$H^*(\eta \circ \omega) = H^*(\omega) \circ H^*(\eta).$$

- If $\mathcal{F}$ is a fusion system on a finite p-group and $\omega \in A_{\mathcal{F}}(S,S)_{(p)}$ is right $\mathcal{F}$-stable with $\epsilon(\omega) \in \mathbb{Z}_{(p)}^{\times}$, then $H^*(\omega)$ is split surjective onto $H^*(S,A)^{\mathcal{F}}$ and is multiplication by $\epsilon(\omega)$ on $H^*(S,A)^{\mathcal{F}}$.
- If $\mathcal{F} = \mathcal{F}_S(G), S \in \mathrm{Syl}_p(G)$, take

$$\omega = [G] = \sum_{x \in [S \backslash G / S]} [S \cap S^x, c_x)]_S^S,$$

then

$$H^*([G]) = \sum_{x \in [S \backslash G / S]} t_{S \cap S^x}^{S} \circ c_x \circ r_{S \cap S^x}^{S} = r_S^G \circ t_S^G.$$

- This shows that the map $H^*(\omega)$ is a generalization of the transfer map in group cohomology, equipped with "S-S-Mackey decomposition". This enables us to generalize Yoshida's theorem to fusion systems "intrinsically", i.e. without resorting to reduction (to finite groups) argument.
- The same strategy almost works for Tate's theorem—while Gagola and Isaacs's argument is almost entirely fusion theoretic, they use one (very easy) group theoretic result in a crucial way:

Proposition

Let G be a finite group, $S, L \leq G$, $G = SL$, $N = S \cap L$. Then we have a bijection

$$L/N \cong G/S$$

induced by inclusion $L \hookrightarrow G$.

In the situation of Proposition, we have an isomorphism of (N, S)-bisets

$$S \times_N L \cong G|_{(S,N)}$$

induced from the multiplication map $(x, y) \in S \times L \mapsto xy \in G$. We have a similar result for characteristic idempotents of fusion systems, which indeed establishes Tate's theorem for fusion systems:

Proposition (Ragnarsson)

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Let $O^p_{\mathcal{F}}(S) \leq N \trianglelefteq S$, and let $\mathcal{F}_N$ be the unique subsystem of $\mathcal{F}$ on N of p-power index. Then

$$[N, \iota_N]_N^S \circ \omega_{\mathcal{F}_N} = \omega_{\mathcal{F}} \circ [N, \iota_N]_N^S$$

Definition

Let S be a p-subgroup of G. Define $\mathcal{F} = \mathcal{F}_S(G)$ to be the category with

- objects: all subgroups of S
- morphisms: $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_G(P, Q)$ for $P, Q \leq S$

Proposition

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then there is a finite group G containing S as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.

Proof.

Let $[X] \in A^+_{\mathcal{F}}(S, S)$ be right $\mathcal{F}$-stable with $\epsilon([X]) = |X|/|S| \not\equiv 0 \mod p$. Then

$$\iota \colon S \to G := \mathrm{Aut}(X|_{(1,S)})$$
$$u \mapsto (x \mapsto ux)$$

is injective, and identifying S with $\iota(S)$, we have $\mathcal{F} = \mathcal{F}_S(G)$. □

Definition

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Define the *exoticity index* of $\mathcal{F}$ to be the number

$$\min\{\log_p |S_0 : S| : S \leq S_0 \in \mathrm{Syl}_p(G) \text{ for some finite group } G \text{ s.t. } \mathcal{F} = \mathcal{F}_S(G)\}.$$

Ambitious Question

Given an exotic fusion system $\mathcal{F}$ on a finite p-group S,

- *What is the exoticity index of $\mathcal{F}$?*
- *What are the finite groups G achieving the exoticity index?*

Modest Goal

Given an exotic fusion system $\mathcal{F}$ on a finite p-group S,

- *Find X minimizing the index $|S_0 : S|$.*
- *Cut down G further to $H = \langle N_G(U) \mid U \leq S\ \mathcal{F}\text{-essential}\rangle$.*
  (Then still $\mathcal{F} = \mathcal{F}_S(H)$.)
- *Analyze H.*

- $\mathcal{F}$ fusion system on a finite p-group S
- $[X] \in A^+_{\mathcal{F}}(S,S)$ right $\mathcal{F}$-stable, $\epsilon([X]) = |X|/|S| \not\equiv 0 \mod p$. Write

$$[X] = \sum_{i \in I} c_i [P_i, \varphi_i]^S_S = \sum_{i \in I} c_i [S \times_{(P_i,\varphi_i)} S], \quad c_i > 0,$$

  where $[P_i, \varphi_i] \neq [P_j, \varphi_j]$ if $i \neq j \in I$.
- $\Omega := \coprod_{i \in I} c_i (S/\varphi_i(P_i)) \Rightarrow |\Omega| = \sum_{i \in I} c_i |S : \varphi(P_i)| = |X|/|S|$.

Proposition

1. $G = S \wr \mathrm{Sym}(\Omega)$.
2. $\iota(S) \subseteq \prod_{i \in I} (S \wr \mathrm{Sym}(S/\varphi(P_i)))^{c_i}$
3. *For $u \in S$, the component of $\iota(u)$ in $\mathrm{Sym}(S/\varphi_i(P_i))$ is $u_{S/\varphi_i(P_i)}$.*

Proof.

Fix $[S/\varphi_i(P_i)] = \{t_{ij}\}$. Then

$$S \times_{(P_i,\varphi_i)} S = \coprod_j t_{ij} \times S \cong \coprod_{S/\varphi_i(P_i)} S \quad \text{as right } S\text{-sets.}$$

Let $u \in S$. Let $\sigma_i(u) = u_{S/\varphi_i(P_i)} \in \mathrm{Sym}(S/\varphi_i(P_i))$. Then $ut_{ij}\varphi_i(P_i) = t_{i\sigma_i(u)(j)}\varphi_i(P_i)$. So

$$u(t_{ij}, x) = (ut_{ij}, x) = (t_{i\sigma_i(u)(j)} t^{-1}_{i\sigma_i(u)(j)} ut_{ij}, x) = (t_{i\sigma_i(u)(j)}, \varphi_i^{-1}(t^{-1}_{i\sigma_i(u)(j)} ut_{ij}), x).$$

Therefore, the i-th component of $\iota(u)$ is

$$(\varphi_i^{-1}(t^{-1}_{i\sigma_i(u)(j)} ut_{ij}), \sigma_i(u)) \in S \wr \mathrm{Sym}(S/\varphi_i(P_i)).$$

□

- $B := S \times \cdots \times S \leq G$ ($|\Omega|$ times) $\Rightarrow G \cong B \rtimes \mathrm{Sym}(\Omega)$, $\overline{G} := G/B \cong \mathrm{Sym}(\Omega)$.
- $\iota(S) \cap B$ is strongly $\mathcal{F}$-closed. In particular, if there is no strongly $\mathcal{F}$-closed subgroup of S, then $\overline{\iota(S)} \cong \iota(S) \cong S$.

- $[X_r] := \sum_{i \in I, |S:P_i| = p^r} c_i [P_i, \varphi_i]_S^S$

Proposition

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Let $[X] \in A^+_{\mathcal{F}}(S,S)$ be right $\mathcal{F}$-stable with $\epsilon([X]) = |X|/|S| \not\equiv 0 \mod p$. Let $\varphi_{i,j} \colon P_i \to P_j$ be the $\mathcal{F}$-maps between subgroup of S of index p with distinct $[P_i, \varphi_{i,j}]$. Then there are integers $c_0 \geq 1$ and $c_1^{(i)} \geq 0$ for each i such that

$$[X_0] = \sum_{\alpha \in \mathrm{Out}_{\mathcal{F}}(S)} c_0 [S, \alpha]_S^S,$$

$$[X_1] = \sum_{\varphi_{i,j} \text{ extendable}} c_1^{(i)} [P_i, \varphi_{i,j}]_S^S + \sum_{\varphi_{i,j} \text{ nonextendable}} (c_0 + p c_1^{(i)}) [P_i, \varphi_{i,j}]_S^S.$$

In particular, there exists $[X]$ with

$$[X_0] = \sum_{\alpha \in \mathrm{Out}_{\mathcal{F}}(S)} [S, \alpha]_S^S, \quad [X_1] = \sum_{\varphi_{i,j} \text{ nonextendable}} [P_i, \varphi_{i,j}]_S^S.$$

Ruiz-Viruel Exotic Systems

- $S \cong p_+^{1+2}$, p an odd prime
- $Z(S) = \langle z \rangle \cong C_p$
- $V_i = \langle z, u_i \rangle \cong C_p \times C_p \quad (0 \le i \le p)$
- $\varphi_{i,j}^{k,l} \colon V_i \to V_j$, $\varphi_{i,j}^{k,l}(z) = u_j^k$, $\varphi_{i,j}^{k,l}(u_i) = z^l$
- $\psi_{i,j}^m \colon \langle u_i \rangle \to \langle u_j \rangle$, $\psi_{i,j}^m(u_i) = u_j^m$
- $\mathcal{F}$ fusion system on S such that all the V_i are $\mathcal{F}$-radical:

$$\mathrm{Aut}_{\mathcal{F}}(V_i) \cong \mathrm{SL}_2(p) : r_i, \quad r_i | (p-1)$$

- $f := \frac{|\mathrm{Out}_{\mathcal{F}}(S)|}{p-1} = |V_i^{\mathcal{F}}| r_i$
  ($\therefore$ Consider the restriction map $\mathrm{Aut}_{\mathcal{F}}(S) \to \mathrm{Hom}_{\mathcal{F}}(V_i, S)$.)

Proposition

Let $\mathcal{F}$ be a fusion system on $S \cong p_+^{1+2}$, p odd, such that all the V_i are $\mathcal{F}$-radical. Then there is a unique minimal right $\mathcal{F}$-stable $[X] \in A_{\mathcal{F}}^{+}(S,S)$ with $|X|/|S| \not\equiv 0 \mod p$ given as follows:

$$[X_0] = \sum_{\alpha \in \mathrm{Out}_{\mathcal{F}}(S)} [S, \alpha]_S^S,$$

$$[X_1] = \sum_{\varphi_{i,j}^{k,l} \in \mathrm{Hom}_{\mathcal{F}}(V_i, V_j)} [V_i, \varphi_{i,j}^{k,l}]_S^S,$$

$$[X_2] = \sum_{\substack{V_i \cong_{\mathcal{F}} V_j \\ 1 \le m \le p-1}} (f - r_j)[\langle u_i \rangle, \psi_{i,j}^m]_S^S + \sum_{\substack{V_i \not\cong_{\mathcal{F}} V_j \\ 1 \le m \le p-1}} f[\langle u_i \rangle, \psi_{i,j}^m]_S^S$$

Note that X involves extendable morphisms. X is also left $\mathcal{F}$-stable.

Question

For arbitrary fusion system $\mathcal{F}$ on a finite p-group S,

- *Is a minimal right $\mathcal{F}$-stable biset X with $|X|/|S| \not\equiv 0 \mod p$ unique?*
- *Is it also left $\mathcal{F}$-stable?*

$\mathrm{Out}_{\mathcal{F}}(S)$	$\lvert V_i^{\mathcal{F}} \rvert$	$\mathrm{Aut}_{\mathcal{F}}(V_i)$	Group	p	f	r_i	e
D_8	$2+2$	$\mathrm{GL}_2(3)$	${}^2F_4(2)'$	3	4	2, 2	$8 \cdot 121$
SD_{16}	4	$\mathrm{GL}_2(3)$	J_4	3	8	2	$16 \cdot 121$
$4S_4$	6	$\mathrm{GL}_2(5)$	Th	5	24	4	$96 \cdot 781$
$6^2 : 2$	$6+2$	$\mathrm{SL}_2(7) : 2$, $\mathrm{GL}_2(7)$		7	12	2, 6	$72 \cdot 2801$
$D_{16} \times 3$	$4+4$	$\mathrm{SL}_2(7) : 2$		7	8	2, 2	$48 \cdot 2801$
$SD_{32} \times 3$	8	$\mathrm{SL}_2(7) : 2$		7	16	2	$96 \cdot 2801$

- $e_i = |X_i|/|S|$, $d_i = e_i/p^i$= no. of S-orbits in Ω of size p^i
- $d_0 = |\mathrm{Out}_{\mathcal{F}}(S)|$
- $d_1 = (p+1)|\mathrm{Out}_{\mathcal{F}}(S)|$
- $d_2 = (p+1)p|\mathrm{Out}_{\mathcal{F}}(S)|$
- $e = |X|/|S| = e_0 + e_1 + e_2 = d_0 + pd_1 + p^2 d_2$
  $= (1 + p + p^2 + p^3 + p^4)|\mathrm{Out}_{\mathcal{F}}(S)| = \frac{p^5-1}{p-1}|\mathrm{Out}_{\mathcal{F}}(S)|$

Action of G on Ω

- If $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, we have

$$[X] \circ [S,\alpha]_S^S = [X] \circ [S,\mathrm{id}]_S^S = [X],$$

and so

$$\sum_{i\in I} c_i[P_i, \varphi_i \circ \alpha] = \sum_{i\in I} c_i[P_i,\varphi_i].$$

For each bijective correspondence between the basis elements appearing in the above equations (counting multiplicities), there is an element $g \in G = \mathrm{Aut}(X|_{(1,S)})$ such that $\psi(u) = gug^{-1}$ for all $u \in Q$ and such that $\overline{g} \in \overline{G}$ preserves that bijective correspondence.

- If $\psi = \varphi_{00}^{rs} \in \mathrm{Aut}_{\mathcal{F}}(V_0)$, we have

$$[X] \circ [V_0,\psi]_{V_0}^S = [X] \circ [V_0, \iota_{V_0}]_{V_0}^S,$$

and so

$$\sum_{i\in I} c_i \sum_{t\in[P_i\backslash S/V_0]} [P_i \cap V_0, \varphi_i \circ c_t \circ \psi]_{V_0}^S = \sum_{i\in I} c_i \sum_{t\in[P_i\backslash S/V_0]} [P_i\cap V_0, \varphi_i \circ c_t]_{V_0}^S.$$

Writing out first few terms, we get

$$\sum_{\alpha \in \mathrm{Out}_{\mathcal{F}}(S)} [V_0, \alpha \circ \psi]^S_{V_0} + \sum_{t \in S/V_0} [V_0, \varphi^{kl}_{ij} \circ c_t \circ \psi]^S_{V_0} + \cdots$$

$$= \sum_{\alpha \in \mathrm{Out}_{\mathcal{F}}(S)} [V_0, \alpha|_{V_0}]^S_{V_0} + \sum_{t \in S/V_0} [V_0, \varphi^{kl}_{ij} \circ c_t|_{V_0}]^S_{V_0} + \cdots .$$

$\alpha|_{V_0}$ and $\varphi^{kl}_{ij} \circ \psi$ are extendable, while $\alpha \circ \psi$ and $\varphi^{kl}_{ij} \circ c_t \circ \psi (t \neq 1)$ are nonextendable. Thus

$$[V_0, \alpha \circ \psi]^S_{V_0} = [V_0, \varphi^{kl}_{ij}]^S_{V_0},$$

$$[V_0, \alpha|_{V_0}]^S_{V_0} = [V_0, \varphi^{kl}_{ij} \circ \psi]^S_{V_0},$$

$$\{[V_0, \varphi^{kl}_{ij} \circ c_t \circ \psi]^S_{V_0}\}_{t \neq 1} = \{[V_0, \varphi^{kl}_{ij} \circ c_t \circ \psi]^S_{V_0}\}_{t \neq 1}.$$

And a similar correspondence can be obtained for the rest of the terms.

In the case $p = 3$, $\mathrm{Out}_{\mathcal{F}}(S) \cong D_8$:

$$(\cdots\cdots\cdots)|\overbrace{(\cdots)\cdots(\cdots)}^{8V_0}\cdots\overbrace{(\cdots)\cdots(\cdots)}^{8V_3}|\overbrace{(\cdots\cdots\cdots)\cdots(\cdots\cdots\cdots)}^{24\langle u_0\rangle}\cdots$$

$$(\cdots\cdots\cdots)|\overbrace{(\cdots)\cdots(\cdots)}^{8V_0}\cdots\overbrace{(\cdots)\cdots(\cdots)}^{8V_3}|\overbrace{(\cdots\cdots\cdots)\cdots(\cdots\cdots\cdots)}^{24\langle u_0\rangle}\cdots$$

$N_G(S)$-action on Ω

$$(\cdots\cdots\cdots)\overbrace{(\cdots)\cdots(\cdots)}^{8V_0}|\cdots\overbrace{(\cdots)\cdots(\cdots)}^{8V_3}\overbrace{(\cdots\cdots\cdots)\cdots(\cdots\cdots\cdots)}^{24\langle u_0\rangle}|\cdots$$

$$(\cdots\cdots\cdots)\overbrace{(\cdots)\cdots(\cdots)}^{8V_0}|\cdots\overbrace{(\cdots)\cdots(\cdots)}^{8V_3}\overbrace{(\cdots\cdots\cdots)\cdots(\cdots\cdots\cdots)}^{24\langle u_0\rangle}|\cdots$$

$N_G(V_0)$-action on Ω

Proposition

Let $\mathcal{F}$ be a fusion system on $S \cong p_+^{1+2}$, p odd, such that all the V_i are $\mathcal{F}$-radical. Let $[X] \in A_{\mathcal{F}}^{+}(S, S)$ be the unique minimal right $\mathcal{F}$-stable element with $|X|/|S| \not\equiv 0 \mod p$. Let $G = \mathrm{Aut}(X|_{(1,S)})$, and $H = \langle N_G(S), N_G(V_i) \mid 0 \leq i \leq p \rangle$. Then $\overline{H}$ is transitive on Ω.

Question

Is $\overline{H}$ primitive?

"Conjecture"

$\overline{H}$ is primitive iff $\mathcal{F}$ has no normal subsystem on S.