GEOMETRIC TOPOLOGY

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Contents

1. Preliminaries

Before looking at particular examples of topological spaces we recall some fundamental concepts in topology. The reader who is fluent with the basics of point set topology may as well skip this subsection.

Many of the spaces we will look at are subspaces of the Euclidean space $\mathbf{R}^n$. Thus the first concept we recall is that of the subspace topology.

Definition 1.1. *Let X be a topological space and let $A \subseteq X$ be a subset. The subspace topology on A induced by the topology of X is the topology where a subset $U \subseteq A$ is open if and only if there exists an open subset $V \subseteq X$ such that $U = V \cap A$.*

Thus for example, let X be the real line $\mathbf{R}$ and let $A = (0, 1]$. Then any interval of the form $(a, 1]$ for $0 \leq a < 1$ is an open subset of A, since

$$(a, 1] = (a, 1.5) \cap A$$

and $(a, 1.5)$ is open in $\mathbf{R}$. Another simple example is given by any continuous curve L in $\mathbf{R}^2$. The curve itself as a subset of $\mathbf{R}^2$ is closed, but open subsets of L with the subspace topology are given by intersections of open subsets of $\mathbf{R}^2$ with L.

Many familiar topological spaces obtain their topology from a metric defined on them. We recall the definition of a metric.

Definition 1.2. *Let X be a set. A function $d : X \times X \longrightarrow \mathbf{R}_{\geq 0}$ is said to be a (Euclidean) metric on X if it satisfies the following axioms*

1. *$d(x, x) = 0$ for all $x \in X$ and $d(x, y) = 0$ if and only if $x = y$.*
2. *The triangle inequality is satisfied namely for any $x, y, z \in X$,*

$$d(x, y) + d(y, z) \geq d(x, z).$$

The most common example of a metric space is of course $\mathbf{R}^n$, where the Euclidean metric is given by

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2},$$

where $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$.

Recall that a basis for the topology on a topological space X is a collection $\mathcal{U}$ of subsets of X such that every open set in X can be obtained by unions and finite intersections of subsets in $\mathcal{U}$.

If X is a metric space then the natural topology on X induced by the metric has a basis given by all open balls in the metric. Namely, all susets of the form

$$B(x, r) = \{\ y \in X \mid d(x, y) < r\ \}.$$

Next we recall what it means for a function $f : X \longrightarrow Y$ to be continuous.

Definition 1.3. *A function $f : X \longrightarrow Y$ between two topological spaces is said to be continuous if whenever $U \subseteq Y$ is an open set, the inverse image $f^{-1}(U) \subseteq X$ is open in X.*

The next concept we need to recall is that of a topological equivalence or homeomorphism of spaces.

Definition 1.4. *Two topological spaces X and Y are said to be homeomorphic or topologically equivalent if there exists a continuous bijection $f : X \longrightarrow Y$ with a continuous inverse function.*

Example 1.5. *Consider the standard n-sphere of radius 1 given as the set of all points $(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1}$ satisfying the equation*

$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1.$$

On the other hand, consider a non standard n-sphere $\bar{S}^n$ given as the set of all solution to the equation

$$a_0 y_0^2 + a_1 y_1^2 + \cdots + a_n y_n^2 = 1,$$

where $a_i \in \mathbf{R}$ are arbitrary positive real numbers. We claim that the S^n and $\bar{S}^n$ are in fact homeomorphic and thus from a topological point of view equivalent. Indeed, define $f : S^n \longrightarrow \bar{S}^n$ by

$$f(x_0, x_1, \ldots, x_n) = (\frac{x_0}{\sqrt{a_0}}, \frac{x_1}{\sqrt{a_1}}, \ldots, \frac{x_n}{\sqrt{a_n}}).$$

It is easy to see that f is continuous and that every point in its image is in fact in $\bar{S}^n$ (justify both claims). To see that it is a homeomorphism we need to show that it is has a continuous inverse. Thus define $g : \bar{S}^n \longrightarrow S^n$ by

$$g(y_0, y_1, \ldots, y_n) = (\sqrt{a_0}y_0, \sqrt{a_1}y_1, \ldots, \sqrt{a_n}y_n).$$

Once more, it is easy to verify that g is continuous and its image is contained in S^n. We leave it for the reader to verify that f and g are mutual inverses. Thus throughout these notes the n-sphere, i.e. any space homeomorphic to the n-sphere of radius 1 about the origin in $\mathbf{R}^{n+1}$, will be denoted by S^n.

Definition 1.6. *An open cover of a topological space X is a collection $\mathcal{U}$ of open subsets of X, such that the union of all of them contains X. A topological space X is said to be compact if any open covering of X has a finite sub covering.*

Example 1.7. *For any $n > 0$, the Euclidean n space $\mathbf{R}^n$ is not compact. The Heine-Borel theorem in $\mathbf{R}^n$ is the statement that every closed and bounded subset of $\mathbf{R}^n$ is compact. Let N denote the "north pole" of the n sphere S^n, i.e. the point $(0, \ldots, 0, 1)$. Then $S^n \setminus N$ is homeomorphic to $\mathbf{R}^n$ and is thus not compact. The sphere S^n by contrast is compact.*

2. Topological Surfaces

Topological surfaces are going to be a main source of examples for this course. In this section we define them and give a rather intuitive discussion of a few examples. In later sections we develope the tools needed to study topological surfaces in greater depth and rigor, A topological surface is, roughly speaking, a space which locally look like a plane. The reader may already be familiar with many surfaces. For example, in advanced calculus one frequently meets surfaces which occur as the set of zeroes of a polynomial equation in three variables in $\mathbf{R}^3$. For instance The 2-dimensional sphere of radius 1 around the origin in $\mathbf{R}^3$, is the set of all solutions of the equation

$$x^2 + y^2 + z^2 = 1.$$

The equation $x^2 + y^2 - z = 0$ gives a paraboloid, which is another example of a surface. Of course, not only polynomial functions define surfaces. The graph of any reasonable function $z = f(x, y)$ is a surface. As these spaces are naturally subsets of $\mathbf{R}^3$, their natural topology is the subspace topology induced from $\mathbf{R}^3$.

The examples above are analytic in nature. Our point of view on topological spaces will not be analytic. Indeed, analytically two spheres of different radii and not the same space. A sphere and an ellipsoid can also be distinguished by their analytic properties. As topological spaces though all spheres, ellipsoids or any other reasonable deformation of a sphere are exactly the same. More precisely, they are homeomorphic.

Definition 2.1. *A topological* surface *is a space S such that every $x \in S$ has a neighbourhood homeomorphic to either $\mathbf{R}^2$ or $\mathbf{R}^2_{\geq 0}$, the upper half plane. The collection of all points x having a neighbourhood homeomorphic to $\mathbf{R}^2$ form $Int(S)$, the* interior *of S. The points $y \in S$ for which every open neighbourhood is homeomorphic to the upper half plane are called* boundary *points and the collection of all of them forms ∂S, the boundary of S. A surface S is said to be* closed *if its boundary is empty.*

Most surfaces which we shall consider will be closed compact surfaces, for which there is a beautiful classification theorem. We have already seen an example of a closed surface, namely the 2-sphere S^2.

Exercise 2.2. *Convince yourself, using the definition above, that S^2 is indeed a closed surface.*

Possibly a bit less familiar than the 2-sphere is the torus T^2, the easiest definition of which is as the product space $S^1 \times S^1$ of two circles. Another way of constructing the torus is by taking a rectangle and identifying parallel edges keeping track of their orientation. The first identification gives a cylinder and the second yields the torus. As we shall see later, any surface can be constructed this way, by identifying edges of certain polyhedra. At this stage the word "identification" used above does not have clear mathematical meaning. We shall give it one soon.

Notice that both the torus, like the 2-sphere, is a closed surface. Notice that both S^2 and T^2 are surfaces with well defined "inside" and "outside" and there is no way to move from one to the other without "tearing" the surface. This may suggest that the notion of a closed surface corresponds to having this more intuitive feature. However, as we shall see later there are surfaces which are closed, but have no “inside” or “outside”.

There are many examples of surfaces with boundary. The easiest way to construct such examples is by removing a disjoint union of open disks from a closed surface. For instance removing a single open disk from the 2-sphere gives a space which is homeomorphic to the closed 2-disk. Removing two open disks from the 2-sphere gives a space homeomorphic to a closed cylinder etc.

Exercise 2.3. *Describe the surfaces given by removing two open disks from a 2 sphere. You should be able to identify it as a familiar surface. Describe the surface oftained by removing an open disk from a torus. If the disk removed is “very large”, what “shape” does the resulting surface have?*

Let us now consider some less familiar examples. We shall be using techniques here, which we shall only study in detail later, but the construction should be intuitively clear. Consider the 2-sphere S^2. It is convenient for the purpose of this example to think about the actual unit 2-sphere about the origin in $\mathbf{R}^3$. Each point $x = (x_1, x_2, x_3)$ on the sphere has an antipodal point, namely the point $-x = (-x_1, -x_2, -x_3)$. Consider the space $\mathbf{R}P^2$, which is obtained from the sphere by identifying each point with its antipodal. It is rather useless, as you will observe quickly, to try and imagine how the resulting space looks like. However, in formal terms it is very easy to describe. Namely, points in $\mathbf{R}P^2$ are in 1-1 correspondence with unordered pairs of points $\{x, -x\}$ where x is a point on the 2-sphere.

This defines $\mathbf{R}P^2$ as a set of points and we shall use the notation $\langle x\rangle$ for such a pair. What we need to do next is to define the topology on it, namely, we should specify which are the open subsets in $\mathbf{R}P^2$. We shall do it here in a rather loose manner and more rigorously later, when we discuss identification spaces.

There is an obvious map $\pi : S^2 \longrightarrow \mathbf{R}P^2$ which takes a point on the sphere to the corresponding pair $\langle x\rangle = \{x, -x\}$ in $\mathbf{R}P^2$. The topology on $\mathbf{R}P^2$ is defined in the most economical way which will make the map π continuous. More specifically, a subset U of $\mathbf{R}P^2$ is open if and only if the subset $\pi^{-1}(U)$ of S^2 is open as a subset of the sphere. It is intuitively clear from the definition that the resulting topological space is a surface without a boundary, although proving it rigourously takes some work. The surface $\mathbf{R}P^2$ is called the real projective plane. An interesting feature it has is that the notion of orientation doesn't make sense on it. This type of surfaces are called non-orientable. We shall make this term precise later on.

You might have an easier time understanding $\mathbf{R}P^2$, using the following construction, which yields a space homeomorphic to the one constructed above. Namely, consider the solid square, consisting of all points in $(x, y) \in \mathbf{R}^2$, such that $-1 \leq x, y \leq 1$. Identify each point on the top edge $(x, 1)$ with the point $(x, -1)$ on the bottom edge (notice that this gives a cylinder). Then identify each point on the left edge $(-1, y)$ with the point $(1, -y)$ on the right edge. This construction gives another way of looking at $\mathbf{R}P^2$, although at this point it might not be clear why this is the case.

Yet another example for a non-orientable surface without a boundary is the so called Klein bottle K^2. The easiest way to describe this surface is using the pasting technique which we used in the alternative description of $\mathbf{R}P^2$. Thus consider the same square as above and identify each point $(x, 1)$ in the top edge with the point $(-x, -1)$ in the bottom edge and each point $(-1, y)$ in the left edge with $(1, -y)$ in the right edge.

Notice that both for $\mathbf{R}P^2$ and for K^2, to do the second identification in 3-space the surface would have to cross itself, thus creating identifications which are not part of its definition. Thus in order to "visualise" the real projective plane and the Klein bottle one really has to think about them as existing in $\mathbf{R}^4$ rather than $\mathbf{R}^3$. Notice that the Klein bottle is a bottle with only one side. Namely, the concept of inside and out do not make sense on it.

The close our introductory discussion of surfaces, let us look at a non-orientable surface with a boundary. Start as in the construction of the Klein bottle, but only do the first identification. The resulting space is called the Möbius band. Notice that its boundary consists of a single circle. Just like the Klein bottle the Möbius band has only one side.

3. Constructing new spaces out of old - Connected Sums

In this section we will consider a standard topological construction,which will allow us to construct new surfaces. This construction is a particular case of a more general one which will be introduced in the following chapter.

We start by an easy example of a much more general technique. Namely let S and T be two surfaces. By definition every point in the interior of a surface has a neighbourhood homeomorphic to an open disk. Thus given S and T, it is possible to remove an open disk from S and another from T and then glue the resulting spaces along the circular boundary components thus created. This operation is called the connected sum of S and T and is denoted $S\#T$.

Although the construction is intuitively clear, it is worthwhile making the definition of connected sums a bit more rigorous. This is done as follows: Let T and S be surfaces as before. Choose any two interior points points $x_S \in S$ and $x_T \in T$ and let D_S and D_T denote open disks in S and T respectively, which are contained entirely in the interior of the respective surfaces. Such disks can clearly be chosen since x_S and x_T were taken to be interior points. Let $\bar{S}$ and $\bar{T}$ denote the surfaces (with boundary) obtained by removing the disks D_S and D_T. Let $f_S : S^1 \longrightarrow \bar{S}$ and $f_T : S^1 \longrightarrow \bar{T}$ be any two choices of homeomorphisms from the circle to the boundaries ∂D_S and ∂D_T respectively. Now, let $S\#T$ denote the surface obtained from $\bar{S}$ and $\bar{T}$ by identifying any point $y' \in \partial D_S$ with the point $y'' \in \partial D_T$ if there exists some point $z \in S^1$ such that $f_S(z) = y'$ and $f_T(z) = y''$.

This makes it clear what $S\#T$ is as a set of points. Namely, the points in $S\#T$ fall into two families

1. points which are interior in either $\bar{S}$ or $\bar{T}$ and
2. unordered pairs $\{y', y''\}$, where $y' \in \partial S$, $y'' \in \partial T$ and there exists some $z \in S^1$, such that $f_S(z) = y'$ and $f_T(z) = y''$.

We now need to define the topology on $S\#T$. The idea behind the definition was used already in the discussion of the real projective plane. Here however the same idea takes a slightly different angle.

For any two spaces X and Y, let $X \amalg Y$ denote the disjoint union of X and Y. In our case there is a function

$$\pi : \bar{S} \amalg \bar{T} \longrightarrow S\#T,$$

given on each component of the disjoint union by the obvious inclusion. The topology on $S\#T$ is defined to be the smallest topology which makes π a continuous map. Specifically, a subset $U \subseteq S\#T$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\bar{S} \amalg \bar{T}$.

Exercise 3.1. *Prove that the operation of taking connected sums is well defined up to homeomorphism, namely that it does not depend on*

1. *which disks are removed from the respective surfaces as long as both are open and contained entirely in the interior and*
2. *the choice of the homeomorphisms f_S and f_T.*

Connected sums allow us to create a larger variety of surfaces by taking an iterated connected sum of surfaces we are already familiar with. For instance, we can create surfaces with as many "holes" as we want by taking an iterated connected sum of tori, or alternatively attaching handles to a sphere. In fact connected sums enables us to construct all surfaces out of 2-sphere, the cylinder, and the Möbius band or alternatively, replace the cylinder by the torus.

We can now start a more rigorous discussion of surfaces. The basic examples or closed surfaces we have so far are

1. the 2-sphere and the torus (closed orientable surfaces) and
2. the projective plane and the Klein Bottle (closed, non-orientable surfaces).

In addition we have a small variety of surfaces with boundary. An important example to keep in mind of a non-orientable surface with a boundary is the Möbius band. The aims of this section are

1. To discuss more complicated surfaces, which one can get by taking connected sums of the surfaces already introduced
2. To introduce the concept of embedding of a surface in Euclidean space and demonstrate that homeomorphic surfaces can appear in different guises, depending on their embeddings
3. To discuss a classification theorem, which at this point we will not be able to prove, characterising all closed surfaces up to homeomorphism and a choice of embedding.

Consider the 2-sphere S^2. One can cut off two open disks out of the sphere and connect a handle, i.e. a cylinder with two circular boundary components to the holes thus created. The resulting space is easily observed to be homeomorphic to the torus. Alternatively, since we are already comfortable with the torus itself, observe that the same result is obtained by taking the connected sum of the torus and the 2-sphere, although doing that just to obtain the torus again might look a bit pointless.

However, there is no reason in both constructions, why we should limit ourselves to carrying it out only once. One can indeed cut off any even number $2n$ of holes in a sphere and connect handles to create a sphere with n-handles. On the other hand it is also possible to use connected sums to cut and clue together any number of tori. Two questions arise immediately.

1. Is a sphere with n handles homeomorphic to a connected sum of n-tori?
2. Once more than one handle is attached to a 2-sphere, there are many possible way to do the attachment. Namely, the handles can be knotted within themselves in highly non-trivial ways. Does the number of handles uniquely determine the topological type of the resulting surface, or do we have to distinguish between surfaces if the handles are knotted in different ways.

If the handles are attached to the sphere so that they are not knotted it is easy to see that a sphere with n handles is homeomorphic to a connected sum of n tori. To do this just imagine that you have the sphere with n handles in your hands and by pulling and stretching it you can bring it to the desired shape without breaking it. This gives an intuitive positive answer to the first question.

The second is a bit more tricky. Namely, it is clear that just by pulling and stretching you can't bring a knotted sphere with handles to the shape of a torus. However, remember that when you visualise this geometric object you are really thinking about it as a subspace of Euclidean 3-space. Clearly the dimension of the space in which the object is embedded does not effect its homeomorphism type. Thus try to imagine the handled sphere as a subspace of $\mathbf{R}^4$. Just as the Klein bottle can be constructed in 4-space without tearing the surface, you can shift handles around in our case to bring the sphere with n handles to the required form of a connected sum of n tori. This shows that knotting of the handles makes no difference to the homeomorphism type of the surface.

The question now becomes, what is the feature of a connected sum of n tori which could distinguish it from a connected sum of $n+1$ tori. You could try to answer an instance of this question in the following.

Exercise 3.2. *Show that a 2-sphere is not homeomorphic to the torus.*

So far we have been discussing orientable closed surfaces. Namely, observe that a sphere with any number of handles has a well defined "inside" and "outside". This is far from being a rigorous definition of orientability, but it will do for now. One can of course consider connected sums of arbitrary surfaces including orientable and non-orientable put together. A fundamental construction corresponding to attaching handles to spheres is removing n disjoint open disks from a sphere and replacing each one of them by a Möbius band. Remember that a Möbius band has only "one side" and in particular its boundary consists of a single circle. Thus this construction makes sense, as long as one thinks about it as occuring in $\mathbf{R}^4$. Examples of the surfaces obtained this way were already given. Namely, for $n = 1$ one gets the projective plane and $n = 2$ gives the Klein Bottle.

To complicate things even more, let us now discuss in some detail the concept of embeddings.

Definition 3.3. *Let $f : X \longrightarrow Y$ be a continuous map. Then f is called an embedding if, considered as a map from X to the image subspace $Im(f) \subseteq Y$, f is a homeomorphism.*

Two subspaces V and W of the same space Y may be just two different embeddings of the same space, say X. As topological spaces V and W are the same space but for some applications one might like to consider the embedding as part of the data. A particularly nice example was already mentioned. Namely, one can embed a sphere with n handles in $\mathbf{R}^3$ in a garden variety of ways, knotting the handles to one's heart content. Nevertheless, the only difference among all those possible embeddings is the embedding itself and not the space under consideration. How does one distinguish two surfaces then? The answer to that will come later when we consider some topological invariants one can use to solve this problem.

We now state a beautiful theorem, without proof at this stage, which gives a complete classification of all surfaces. To the reader whose imagination has been fired by what appears to be a fantastic variety of different surfaces one can construct, this theorem should be quite surprising.

Theorem 3.4. *Any closed surface is homeomorphic either to a sphere, or to a sphere with a finite number of handles attached to it, or to a sphere with a finite numbers of discs removed and replaced by Möbius bands. Up to homeomorphism every surface arises this way and no two among these are homeomorphic.*

A curious point about this theorem appears to be the assertion that mixed types are not allowed in the classification. Indeed it can be showed for example that a sphere with one handle where a small disk was removed and replaced by a Möbius band is homeomorphic to a sphere with three disjoint disks removed and replaced by Möbius bands.

A sphere with n handles attached is called *a closed orientable surface of genus* n. The genus, roughly speaking, is a measure of the numbers of "holes" or handles in the surface. orientability was already discussed intuitively. We closed this section by giving it a bit more rigorous treatment.

Let S be a surface considered as a subspace of $\mathbf{R}^n$ for some n. Let ω be a differentiable closed curve on S. In analytic terms, if the curve is parametrised as a differentiable function

$$\omega : [0,1] \longrightarrow S \subseteq \mathbf{R}^n,$$

then ω has components $(\omega_1, \dots, \omega_n)$, with $\omega_i : [0,1] \longrightarrow \mathbf{R}$ a differentiable function for each i. Then the derivative ω' is again a function from $[0,1]$ to $\mathbf{R}^n$ given by $\omega' = (\omega'_1, \dots, \omega'_n)$ and for each $t \in [0,1]$, $\omega'(t)$ is a tangent vector to $\omega(t)$.

Consider the tangent plane to the surface and at some point $\omega(t)$ on the curve ω. This plane can be obtained as the linear span of the tangent vector $\omega'(t)$ and the tangent vector to a different curve, say α at the same point. Choose a normal vector $\mathbf{n}$ at the point $\omega(t)$, i.e. a unit vector perpendicular to the tangent plane at the chosen point. Now consider the system given by $\omega'(t)$ and $\mathbf{n}$ at this point. One can think of moving this system along the curve continuously. That is to say, fix the direction on the curve in which you are moving and don't "flip" the normal vector to the "other side". If by the time you get back to the starting point you have the same system you started with and this property holds for every smooth closed curve on the surface, then the surface is called orientable. Otherwise it is called non-orientable.

Exercise 3.5. *Convince yourself that a closed orientable surface of any genus is indeed orientable, with respect to the above definition of orientability, and that the Projective plane and the Klein Bottle are not orientable.*

4. Partitions and Identification Spaces.

In the previous sections we discussed two types of topological constructions. The first type is spaces obtained by glueing two or more spaces together. An example of this type is given by the construction of connected sums of two or more surfaces. The other type is the construction of the real projective plane $\mathbf{R}P^2$ by identifying pairs of antipodal points on the 2-sphere. Although the two constructions appear to be unrelated they are in fact two instances of the same. Indeed, in both cases what one looks at is a space obtained by identifying certain points in a given space. In this section and the next one we shall put these two examples and many more together in a more general context.

Let X be a topological space and let $\mathcal{P}$ denote a partition of X. That is to say $\mathcal{P}$ is a family of non-empty disjoint subsets of X whose union is X itself. For instance if G is a group operating on a set X, then a partition on X is given by considering the orbits of the action. Namely, two points $x, y \in X$ are in the same subset of the partition if and only if there exists an element $g \in G$ such that $g \cdot x = y$. A partition of a pair of surfaces, each with one boundary component homeomorphic to a circle may be given by considering each internal point separately and the boundary points in pairs determined by some homeomorphism.

Given a partition $\mathcal{P}$ of X, define the *identification space* of X with respect to $\mathcal{P}$ to be the space Y obtained by identifying each subset in $\mathcal{P}$ to a single point. There is an obvious function

$$\pi : X \longrightarrow Y,$$

given by sending a point to the point in the identification space given by the subset to which it belongs. Recall that an equivalence relation on the X as a set amounts to specifying a partition of X, namely given a partition of X, two points x and y in X are equivalent if and only if they belong to the same subset in the partition. With this in mind, for a space X, a point $x \in X$ and a partition $\mathcal{P}$ on X, we may denote by $\langle x \rangle$ the class of x with respect to the partition or the corresponding point of the identification space. Thus we may write $\pi(x) = \langle x \rangle$.

The natural topology on the identification space is defined by means of the projection $\pi : X \longrightarrow Y$. Thus a subset U of Y will be called open if and only if $\pi^{-1}(U)$ is open in X. This is called the identification topology and is the largest topology for which the identification map is continuous.

Theorem 4.1. *Let Y be an identification space of X. Then a map $f : Y \longrightarrow Z$ is continuous if and only if the composition $f\pi$ is continuous.*

Proof. The only if part is obvious. For the if part, let $U \subseteq Z$ be an open subset. Then $(f\pi)^{-1}(U)$ is open in X. But

$$(f\pi)^{-1}(U) = \pi^{-1}(f^{-1}(U)),$$

which means means exactly that $f^{-1}(U)$ is open in Y, so f is continuous. □

An identification space can be characterised in a different way. Namely let $f : X \longrightarrow Y$ be an onto function. Suppose that the topology on Y is the largest such that the map f is continuous. Then Y may be called an identification space. For consider the partition of X given by considering the disjoint subsets $f^{-1}(y)$ for all $y \in Y$. Let $\bar{Y}$ be the identification space obtained from X with respect to this partition. Then it is immediate that $\bar{Y}$ and Y are homeomorphic.

Theorem 4.2. *Let $f : X \longrightarrow Y$ be a continuous onto map. Assume in addition that f is either an open map or a closed map. Then f is an identification map.*

Proof. By the remarks above it suffices to show that the topology on Y is the largest such that the map f is continuous. Assume f is an open mapping. Let U be a subset of Y such that $f^{-1}(U)$ is open in X. We must show that U is open in Y. But since f is open it follows that $f(f^{-1}(U)$ is open in Y. Since f is onto one has $f(f^{-1}(U) = U$ and the results follows. The argument in the case that f is a closed mapping is similar. □

Corollary 4.3. *Let $f : X \longrightarrow Y$ be an onto map from a compact space X To a Hausdorff space Y. Then f is an identification map.*

Proof. Any closed subset of a compact space is compact. The image of a compact space under a continuous map is compact and a compact subset of a Hausdorff space is closed. It follows that f is a closed mapping. Hence by the theorem it is an identification map. □

Pasting Spaces Together.

In previous sections we have seen several examples of spaces obtained by pasting together other spaces. For instance pasting cylinders or Möbius bands along a sphere where an appropriate number of disjoint open disks has been removed yields closed topological surfaces, and the classification theorem for such surfarces states that in fact all closed compact surfaces can be obtained this way. We now identify some of the examples already given and present some new ones in the context of identification spaces. In each case we describe the partition such that the respective identification space gives the example under consideration.

The Möbius band. Consider the unit square $[0, 1] \times [0, 1]$. The partition giving rise to the Möbius band consists of the following subsets:

- all singletons $\{(x, y)\}$, one for each point (x, y) such that $0 < x < 1$.
- all pairs of the form $\{(0, y), (1, 1 - y)\}$.

The n-sphere. Consider the unit n-ball D^n and its boundary S^{n-1}. The identification space obtained from D^n by identifying its boundary to a single point is homeomorphic to the n-sphere.

If X is a space and A is a subspace of X then the identification space obtained from X by identifying A to a single point is denoted X/A. Thus for instance we may write $S^n \cong D^n/S^{n-1}$.

The torus. Consider again the unit square as above. The partition which gives the torus consists of the following subsets:

- all singletons $\{x, y)\}$, where the point (x, y) is in the interior of the square.
- all pairs of the form $\{(0, y), (1, y)\}$.
- all pairs of the form $\{(x, 0), (x, 1)\}$.

Exercise 4.4. *Recall that $\mathbf{R}P^2$ can be obtained from a square by identifying each pair of parallel edges in opposite orientation. Follow the lines of the examples above to give the partition on the unit square which gives rise to the real projective plane.*

Exercise 4.5. *The Klein bottle can be obtained from a square by identifying one pair of parallel edges in the same orientation and the other in opposite orientation. Repeat Exercise 4.4 for the Klein bottle.*

Connected sums. Let S and T be two closed surfaces and let $\bar{S}$ and $\bar{T}$ denote the spaces obtained from S and T by removing an open disk off each. Since ∂S and ∂T are homeomorphic, we may choose a homeomorphism $f : \partial S \longrightarrow \partial T$. The partition which gives rise to the connected sum of S and T is the following.

- all singletons $\{z\}$, where z is an interior point in either $\bar{S}$ or $\bar{T}$.
- all pairs $\{y, f(y)\}$, where $y \in \partial T$.

The cone construction. Let X be an arbitrary space and let I denote the unit interval $[0, 1]$. The cone construction CX, is the identification space of the space $X \times I$ obtained from the following partition

- all singletons $\{x, t)\}$ for $t > 0$.
- the subset $\{(x, 1) \mid x \in X\}$.

Thus CX is the space obtained from $X \times I$ by identifying all points $(x, 1)$ to a single point.

Exercise 4.6. *Suppose the X is a subspace of $\mathbf{R}^n$ and let $v \in \mathbf{R}^n$ be a point not in X . Define the geometric cone on X with vertex v to be the space consisting of all points of the for $tv + (1 - t)x$ for all x in X. Show that the geometric cone on X is homeomorphic to CX.*

Now that the reader is a bit more comfortable with the terminology we will simplify it a bit. Namely we will say, if possible, that the identification space Y is obtained from X by identifying certain points in Y, possibly without specifying the partition explicitly.

The suspension construction. Let X be an arbitrary space. Define the suspension of X denoted SX to be the space obtained from a disjoint union of two copies of CX by identifying each point of the form $(x, 1)$ in one copy with its counterpart in the other.

Glueing Lemma. Let X and Y be subspaces of a given space W and let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be continuous functions which agree on the intersection $X \cap Y$. Then one may define

$$(f \cup g) : X \cup Y \longrightarrow Z,$$

by

$$(f \cup g)(w) = \begin{cases} f(w) & \text{if } w \in X \\ g(w) & \text{if } w \in Y. \end{cases}$$

What isn't quite so obvious is that the resulting map is continuous.

Lemma 4.7. *Let $X, Y \subseteq W$ be subspaces and let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be continuous functions which agree on the intersection $X \cap Y$. If X and Y are both closed in $X \cup Y$ then so is $f \cup g$.*

Proof. Let C be a closed subset of Z. Then $(f \cup g)^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ and it suffices to show that $f^{-1}(C)$ is closed in $X \cup Y$ and similarly for $g^{-1}(C)$. We prove the statement for $f^{-1}(C)$. First, recall that X is assumed to be a subspace of W and so its topology is induced from that of W. By continuity of f, the subset $f^{-1}(C)$ is closed in X. By definition of the induced topology this means that there is a closed subset U of W such that $U \cap X = f^{-1}(C)$. Since X is closed in the union $X \cup Y$ and the topology on the union is again induced by the topology on W, there is a closed subset V of W such that $V \cap (X \cup Y) = X$. Finally $U \cap V$ is closed in W and

$$(U \cap V) \cap (X \cup Y) = U \cap (V \cap (X \cup Y)) = U \cap X = f^{-1}(C)$$

so $f^{-1}(C)$ is closed in $X \cup Y$ as well. This completes the proof. □

Exercise 4.8. *Prove the same statement under the assumption that both X and Y are open in the union.*

The pushout construction. The pushout construction is one of the most common and useful constructions in topology. It can best be described as glueing together two spaces along a common subspace. For instance, the construction of connected sums of two surfaces can be thought of as an instance of the pushout construction.

First recall that a map $f : A \longrightarrow X$ is called an embedding if it is a homeomorphism into its image. The map f is called a closed embedding if in addition $f(A)$ is a closed subspace of X.

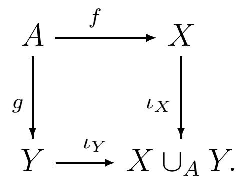
Let X and Y be spaces and suppose $f : A \longrightarrow X$ and $g : A \longrightarrow Y$ are closed embeddings. Define a partition on the disjoint union $X \amalg Y$ by taking

- all singletons $\{x\}$ for $x \in X \setminus f(A)$
- all singletons $\{y\}$ for $y \in Y \setminus g(A)$
- all pairs $\{f(a), g(a)\}$ for all $a \in A$.

Let $X \cup_A Y$ denote the resulting identification space. This space is called the pushout of the system

$$X \xleftarrow{f} A \xrightarrow{g} Y.$$

The inclusions from X and Y into $X \cup_A Y$ gives rise to a commutative square of spaces and maps



The pushout construction is characterized by a universal property, which is the contents of the following

Proposition 4.9. *Let X, Y, A, f and g be as above. Let $\iota_X : X \longrightarrow X \cup_A Y$ and $\iota_Y : Y \longrightarrow X \cup_A Y$ be the inclusion maps. Let $s : X \longrightarrow Z$ and $t : Y \longrightarrow Z$ be continuous maps, such that $f \circ s = g \circ t$. Then there exists a unique continuous map $h : X \cup_A Y \longrightarrow Z$, such that $h \circ \iota_X = s$ and $h \circ \iota_Y = t$.*

Proof. The space $X \cup_A Y$ can be thought of as a union of X and Y intersecting on A, where A is identified with the subset of pairs $\{f(a), g(a)\}$ for all $a \in A$ in the identification space.

Notice that the space $X \cup_A Y$ is an identification space of the disjoint union $X \amalg Y$ and there is an identification map

$$\iota_X \amalg \iota_Y : X \amalg Y \longrightarrow X \cup_A Y.$$

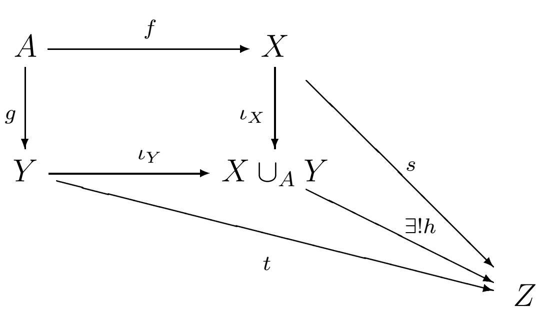
The complement of the subspace $\iota_X(X) \cong X$ in $X \cup_A Y$ is homeomorphic to $Y \setminus g(A)$. Since g is assumed to be a closed embedding, the subspace $g(A) \subseteq Y$ is closed in Y and so $Y \setminus g(A)$ is open in Y. Hence $\iota_X(X)$ is closed in $X \cup_A Y$. Similarly $\iota_Y(Y)$ is closed in $X \cup_A Y$.

The conditions of the Glueing Lemma are now satisfied, where one identifies X with $\iota_X(X)$, Y with $\iota_Y(Y)$ and A with the subset of pairs $\{f(a), g(a)\}$, in $W := X \cup_A Y$. The assumption of the proposition is exactly that s and t agree on the intersection of X and Y, namely A. Thus the glueing lemma gives the required continuous map

$$h : X \cup_A Y \longrightarrow Z.$$

The proof of uniqueness is immediate and is left for the reader. ☐

The universal property of a pushout can be illustrated by the following diagram



Another example of a family of identification spaces will be discussed in the next section.

5. Topological Groups, group actions and identification spaces

Topological groups is another rich source of interesting spaces. A topological group if a topological space G together with a continuous map

$$\mu : G \times G \longrightarrow G$$

satisfying the usual axioms for multiplication in a group (associativity, existence of unit and existence of inverses). Topological groups form a particularly nice family of spaces, as it turns out that the group structure imposes severe restrictions on the topology. There are many natural examples of topological groups. We will mention a few here.

The real and complex numbers. The real and complex numbers are examples of topological fields (and I leave it to you to figure out what that should mean). Thus each one of them has two underlying topological groups; the additive group and the multiplicative group. The additive group of the real numbers $\mathbf{R}^+$ is a connected topological group. The multiplicative group $\mathbf{R}^*$ on the other hand forms a non-connected topological group. Both the additive and the multiplicative groups of the complex numbers are connected.

The circle and the k-Torus. The circle is a subgroup of the multiplicative group of the complex numbers $\mathbf{C}^*$. Indeed it is the subgroup (as well as the subspace) given by

$$S^1 = \{z \in \mathbf{C} \mid |z| = 1\}.$$

Notice that S^1 is a subgroup by virtue of the identities $|z_1 z_2| = |z_1||z_2|$ and $|z^{-1}| = |z|^{-1}$. The topology on S^1 is of course the subspace topology. A Cartesian product of k circles - the k dimensional torus

$$T^k = S^1 \times S^1 \times \cdots \times S^1$$

k times - is also a topological group in a natural way. Namely, the product is given by coordinate wise multiplication

$$(x_1, x_2, \ldots , x_k) \cdot (y_1, y_2, \ldots , y_k) = (x_1y_1, x_2y_2, \ldots , x_ky_k).$$

The topology on T^k is the product topology or equivalently the subspace topology where T^k is considered as a subspace of $\mathbf{C}^k$.

Groups of transformations. Let $\mathbf{F}$ be a topological field (i.e. a topological space $\mathbf{F}$ with two operations $a, m : \mathbf{F} \times \mathbf{F} \longrightarrow \mathbf{F}$, called addition and multiplication, satisfying the usual axioms, which are continuous as maps of spaces. Let $M_n(\mathbf{F})$ denote the set of all linear transformations from $\mathbf{F}^n$ to itself. A choice of basis enables us to identify $M_n(\mathbf{F})$ with the set of all $n \times n$ matrices over $\mathbf{F}$. Thus $M_n(\mathbf{F})$ can be identified with the n^2-dimensional vector spaces $\cong \mathbf{F}^{n^2}$ and hence becomes a topological space by giving it the product topology induced from the topology of $\mathbf{F}$ itself. Furthermore, multiplication and addition of matrices are continuous operations, since they only involve addition and multiplication in $\mathbf{F}$ and so they turn $M_n(\mathbf{F})$ into a topological ring.

Exercise 5.1. *What is $M_1(\mathbf{F})$? Can you see immediately that $M_1(\mathbf{F})$ is a topological ring? What about $M_2(\mathbf{F})$? Try to convince yourself, using the line of argument sketched above that addition of matrices and multiplication of matrices in $M_2(\mathbf{F})$ are continuous operations.*

Now consider the subspace $GL_n(\mathbf{F})$ given by all invertible matrices. The topology on $GL_n(\mathbf{F})$ is defined to be the subspace topology. This subspace is not a ring anymore, since addition of matrices does not preserve invertibility, but rather it is a group with respect to matrix multiplication and is called the general linear group of rank n over $\mathbf{F}$. Within $GL_n(\mathbf{F})$ there are certain subgroups of particular importance. One such subgroup is the special linear group $SL_n(\mathbf{F})$ of all matrices of determinant 1. Let us now specialise to specific fields.

Consider first the case $\mathbf{F} = \mathbf{R}$. There are subgroups

$$SO(n) \subseteq SL_n(\mathbf{R}) \quad \text{and} \quad O(n) \subseteq GL_n(\mathbf{R}).$$

The subgroup $O(n)$ is called the orthogonal group and consists of all transformations which take any orthonormal basis for $\mathbf{R}^n$ to another orthonormal basis, or equivalently, the subgroup of all matrices whose columns form an orthonormal basis for $\mathbf{R}^n$. It is an easy exercise in linear algebra to show that the determinant of all matrices in $O(n)$ is ± 1. The special orthogonal group $SO(n)$ is the subgroup of $O(n)$ of all matrices of determinant 1.

Exercise 5.2. *Show that $O(1)$ is isomorphic to the cyclic group of order 2 and that $SO(2)$ is isomorphic as a topological group to the circle group S^1. Conclude that $O(2)$ is homeomorphic as a space to $SO(2) \times O(1)$, but that as a group it is not isomorphic to it.*

Replacing $\mathbf{R}$ by the complex numbers $\mathbf{C}$ we get the unitary groups $U(n)$ and the special unitary groups $SU(n)$, which is the subgroup of $U(n)$ of matrices with determinant 1. The determinant of every matrix in $U(n)$ is of absolute value 1 just as before, but in the complex case this means that it is a complex number on the unit circle.

Exercise 5.3. *Show that $U(1)$ is isomorphic to the circle group S^1 as a topological group.*

It can be shown that $SO(n)$ and $SU(n)$ are all connected spaces. Also $U(n)$ can be shown to be homeomorphic as a space (not a topological group) to the product space $SU(n) \times S^1$ and so it is connected as well.

Exercise 5.4. *Show that $O(n)$ has two connected components both homeomorphic to $SO(n)$ for every n.*

Another feature of the groups $U(n)$ and $O(n)$ is that they are compact. This also apply to the subgroups $SU(n)$ and $SO(n)$. We closed this preliminary discussion of topological groups by showing that $O(n)$ is compact. The corresponding statement for $U(n)$ is left as an exercise for the reader.

Proposition 5.5. *For every $n \geq 1$ the group $O(n)$ is compact.*

Proof. First notice that $O(n)$ is by definition the group of all $n \times n$ matrices, whose columns form an orthonormal basis for $\mathbf{R}^n$. Thus each column vector corresponds uniquely to a point in the sphere S^{n-1}, considered as a subspace of $\mathbf{R}^n$. This allows an identification of $O(n)$ as a subspace of the n-fold Cartesian product

$$(S^{n-1})^n = S^{n-1} \times S^{n-1} \times \cdots \times S^{n-1}$$

n times. Notice that a finite Cartesian product of spheres is compact because the sphere itself is. Since a closed subspace of a compact space is itself compact, it is enough to show that $O(n)$ is a closed subspace of $(S^{n-1})^n$ or equivalently that it is a closed subset of $M_n(\mathbf{R})$.

Let (,) denote inner product in $\mathbf{R}^n$. Define a map

$$D : M_n(\mathbf{R}) \to \mathbb{R},$$

as follows: For a matrix $A = (u_1, \cdots, u_n)$, where the u_i are column vectors, define

$$D(A) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} |(u_i, u_j)|,$$

where | | means absolute value. Since $(a, b) = 0$ if and only if a and b are orthogonal, it follows that $D(A) = 0$ if and only if every vector in A is orthogonal to all the rest, thus if and only if the columns of A form an orthogonal basis for $\mathbf{R}^n$. Clearly D is a continous function and so $D^{-1}(0)$ is a closed subset of $M_n(\mathbf{R})$.

Now, $O(n)$ is defined as the set of all matrices in $M_n(\mathbf{R})$ whose columns form an orthonormal basis for $\mathbf{R}^n$. Thus

$$O(n) = D^{-1}(0) \cap (S^{n-1})^n,$$

which is obviously closed in $M_n(\mathbf{R})$, being the intersection of two closed subsets. The claim follows. □

Group Actions and Orbit Spaces.

Let X be a topological space and let G be a topological group (possibly discrete). We say that the group G acts on X or that X is a (left) G-space if there is a continuous map

$$\mu : G \times X \longrightarrow X$$

which defines an action of G on X taken as a set of points (Namely only continuity is the additional requirement). Specifically the map μ is required to satisfy

1. $\mu(1, x) = x$ for all $x \in X$
2. $\mu(g, \mu(h, x)) = \mu(gh, x)$ for all $g, h \in G$ and $x \in X$.

Given an action of G on X one can consider the orbit space denoted X/G, which consists of all orbits of the G-action in X. That is to say, for each orbit

$$Gx := \{\mu(g, x) \mid g \in G\} \subseteq X$$

of a point $x \in X$ under the G-action, the orbit space X/G has a single point, denoted $\langle x \rangle$.

Exercise 5.6. *Let X be a G space for some topological group G. Show that the relation on points of X given by $x \sim y$ if x and y are in the same orbit of the G-action on X is an equivalence relation on X and thus divides it into disjoint equivalence classes.*

There is a map $\pi : X \longrightarrow X/G$, which takes every $x \in X$ to the orbit $\langle x \rangle$. The topology of X/G is defined in a way that makes the map π continuous. Namely, a subset U of X/G is open if and only if its inverse image $\pi^{-1}(U)$ is open in X. This construction involves certain subtleties, namely, the topology on the orbit space X/G may be pathological even if one starts with a very nice space X. We shall discuss this point later. For now let us consider some examples.

The real projective space. Consider the n-sphere S^N. The antipodal map defines an action of $\mathbf{Z}/2\mathbf{Z}$, the cyclic group of order 2 on S^n. Namely, if $A : S^n \longrightarrow S^n$ denotes the antipodal map and $\sigma \in \mathbf{Z}/2\mathbf{Z}$ is the non-trivial element define the group action map μ by

$$\mu(\sigma, x) = A(x)$$

for any $x \in S^n$. Explicitely, The unit n-sphere is the subspace of $\mathbf{R}^{n+1}$ consisting of all points $(x_1, \cdots x_{n+1})$ such that

$$x_1^2 + \cdots x_{n+1}^2 = 1.$$

Thus there is an antipodal map $A : S^n \longrightarrow S^n$ given by

$$A(x_1, \cdots x_{n+1}) = -(x_1, \cdots x_{n+1}).$$

As before this map defines an action of the group $\mathbf{Z}/2\mathbf{Z}$ on the n-sphere and the resulting orbit space is called the real projective n-space and is denoted $\mathbf{R}P^n$.

The reader should look at the definition of the projective plane given above to verify that it coincides with what we now call the orbit space of S^2 under the action of $\mathbf{Z}/2\mathbf{Z}$. This setup however gives a way of defining the real projective space for arbitrary $n > 0$.

The space R/Z. Consider the action of the additive group of integers $\mathbf{Z}$ on the real line $\mathbf{R}$, given by $n \cdot x = n + x$. The orbit space is the identification space of $\mathbf{R}$ where we identify two points if they differ by an integer. We claim that the resulting identification space is homeomorphic to the circle S^1. Indeed consider the exponential map $exp : \mathbf{R} \longrightarrow S^1$ given by $exp(x) = e^{2\pi i x}$. If $x - y \in \mathbf{Z}$, namely if $\langle x \rangle = \langle y \rangle$ in $\mathbf{R}/\mathbf{Z}$ then $exp(x) = exp(y)$. Thus exp induces a well defined map $e : \mathbf{R}/\mathbf{Z} \longrightarrow S^1$, given by $e\langle x \rangle = e^{2\pi i x}$. Since $e \circ \pi = exp$ it follows at once that e is continuous. It is obviously onto, because exp is onto. To see that it is $1-1$, notice that in fact $x - y \in \mathbf{Z}$ if and only if $exp(x) = exp(y)$. Thus $e\langle x\rangle = e\langle y\rangle$ if and only if $\langle x\rangle = \langle y\rangle$, which means that e is $1-1$. The proof that e is an open mapping (and thus a homeomorphism) can be carried out as follows. Notice first that exp is an open mapping, namely it carries open subsets of the real line to open subsets of the circle. A subset $U \subseteq \mathbf{R}/\mathbf{Z}$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbf{R}$. But $e(U) = exp(\pi^{-1}(U))$ is open in S^1, whence the claim.

The torus. Consider the action of the additive group $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{R}^2$, given simply by taking the product of the action of $\mathbf{Z}$ on $\mathbf{R}$ with itself. Explicitely,

$$(m, n) \cdot (x, y) = (n + x, m + y).$$

The resulting orbit space is, not surprisingly, $S^1 \times S^1$, i.e. the 2-torus. The $n-torus$, $T^n = S^1 \times S^1 \times \cdots \times S^1$ n-times is obtained in a similar way from the action of $\mathbf{Z}^n$ on $\mathbf{R}^n$.

Notice that we have not included a proof that $\mathbf{R} \times \mathbf{R}/\mathbf{Z} \times \mathbf{Z}$ is homeomorphic for $S^1 \times S^1$. The proof of the corresponding general statement below is easy and is left to the reader.

Exercise 5.7. *Let X' be a G'-space and let X'' be a G''-spaces. Then the product group $G' \times G''$ acts on $X' \times X''$ and there is a homeomorphism*

$$(X' \times X'')/(G' \times G'') \cong X'/G' \times X''/G''.$$

Before we give our next example, we recall another concept about group actions. Let X be a G-space. We say that G acts transitively on X if for every $x, y \in X$ there is some $g \in G$ such that $x = gy$. It is easy to check the following statement.

Exercise 5.8. *A group G acts transitively on a set X if and only if the action has only one orbit.*

The space $S^{n-1}/O(n)$. The orthogonal group $O(n)$ acts on $\mathbf{R}^n$ and its action preserves the length of vectors. That is to say, any orthonormal basis for $\mathbf{R}^n$ is carried by an element of the orthogonal group to another orthonormal basis. Thus the action induces an action of $O(n)$ on the unit sphere S^{n-1} in $\mathbf{R}^n$.

Claim: The action of $O(n)$ on the sphere is transitive.

We must show that for every two points x and y in S^{n-1} there is some $A \in O(n)$ such that $Ax = y$. To see this consider x and y as unit vectors in $\mathbf{R}^n$. Let U_x and U_y denote the orthogonal complements of x and y respectively. Choose orthonormal bases $u_1, \cdots u_{n-1}$ for U_x and $v_1, \cdots v_{n-1}$ for U_y. Then there is a linear transformation A which takes x to y and u_j to v_j. Since A takes an orthonormal basis to another orthonormal basis, it is an orthogonal transformation, i.e. an element of $O(n)$. $\square$

A consequence of the above discussion is that the orbit space $S^{n-1}/O(n)$ is a single point.

Coset spaces. Let G be a topological group and H a topological subgroup. Then H acts on G by left translation, i.e. the action map $\mu : H \times G \longrightarrow G$ is the restriction of the multiplication map μ on $G \times G$ to the subspace $H \times G$, which is thus obviously continuous. The orbits of this action are exactly the left cosets of H in G. The topology is defined as usual in an orbit space, but has a nicer more explicit description in this case. Namely let $U \subseteq G/H$ be a subset. Let $\bar{U}$ denote the subset of G given by the union of all left cosets gH with $\langle g \rangle \in U$. Then U is open in G/H if and only if $\bar{U}$ is open in G. As an example we have the following.

The spaces $O(m+k)/O(k)$. Let $\mathbf{e}_1, \cdots, \mathbf{e}_{m+k}$ denote the standard basis vectors in $\mathbf{R}^{m+k}$. Then a matrix $A \in O(m+k)$ sends $\mathbf{e}_i$ for all $m+1 \leq i \leq m+k$to itself if and only if it has the form $\left(\begin{smallmatrix} A & 0 \\ 0 & I_k \end{smallmatrix}\right)$, where A is a matrix in $O(m)$ and I_k is the identity matrix in $O(k)$. This defines an embeding of the group $O(k)$ in $O(m+k)$.

The coset spaces $O(m+k)/O(k)$ are classically denoted $V_{m+k,k}$ and are called the Stiefel manifolds. They are spaces of fundamental importance in topology. These spaces have some very interesting features, which at this point we don't have enough tools to analyze. However, we observe next that there is one case with which we are already very familiar, namely the case where $m = 1$ and $k = n-1$.

Define a map $\pi : O(n) \longrightarrow S^{n-1}$ by $\pi(A) = A(\mathbf{e}_n)$. Then π is clearly onto (Why? See the explanation above about transitivity of the action of $O(n)$ on S^{n-1}). Since $O(n)$ is compact and S^{n-1} is Hausdorff, it follows that π is an identification map. The inverse image of a point $x \in S^{n-1}$ is the set of all matrices in $O(n)$ taking $\mathbf{e}_n$ to x. But if A is such a matrix then every other such matrix B can be written in the form AU where U is a matrix in the image of $O(n-1)$ under the embeding above. That is to say the difference $U = A^{-1}B$ is a matrix which takes e_n to itself and hence is in the image of $O(n-1)$. Thus the inverse image of any point x on the sphere is the left coset of A in $O(n)$ modulo $O(n-1)$, where A is an arbitrary matrix in $O(n)$ which takes e_n to x. Since this holds for every point on the sphere, it follows that the partition on $O(n)$ defining the identification space given by S^{n-1} is the same as the partition given by the left cosets of $O(n-1)$ in $O(n)$. As a consequence we obtain that $O(n)/O(n-1) \cong S^{n-1}$.

6. Homotopy and Homotopy Type

So far we have been discussing topological spaces under the assumption that two spaces may be identified if their topological type is the same, namely, when they are homeomorphic. The topological type of a space however is a very strong invariant of the space in the sense that for many applications two spaces may be thought of as the same although they are not homeomorphic. For these applications it is many times useful to think about two spaces as identical if one can continuously deformed into the other. For example, any Euclidean space can be deformed continuously into a single point. However the circle or any sphere or the torus cannot be deformed to a point. It is also not hard to see that the circle cannot be deformed into a 2-sphere and generally an n-sphere cannot be deformed into a k-sphere if $n \neq k$. This motivates the idea of homotopy and homotopy type. Whereas homotopy is an equivalence relation between maps, homotopy type is an equivalence relation between spaces. The study of homotopy, homotopy types and features of topological spaces which are preserved under these relations forms a rich, well developed mathematical discipline called homotopy theory. In this section we describe some of the fundamental concepts of homotopy theory with an eye towards geometric applications.

Homotopy of Maps. Let I denote the unit interval $[0, 1]$.

Definition 6.1. *Let $f, g : X \longrightarrow Y$ be two maps. We say that f and g are homotopic if there exists a continuous function*

$$H : X \times I \longrightarrow Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

We will somtimes call a map $H : X \times I \longrightarrow Y$ a homotopy without specifying the two maps which it relates. To make sense of this, consider the notation $H_t : X \longrightarrow Y$ for the continuous map $H(x, t)$, where $t \in I$ is kept fixed. The homotopy H relates the maps H_0 and H_1 and indeed any map H_t in between the two. Furthermore, there is a way to define a topology on the set $\mathrm{Map}(X, Y)$ of all continuous maps from X to Y. Thinking about this set as a topological space, a homotopy from f to g is simply a continuous path in $\mathrm{Map}(X, Y)$ starting at f and ending at g.

Example 6.2. *Let C be a convex subset of $\mathbf{R}^n$ and let $f, g : X \longrightarrow C$ be any two maps. By definition of convexity the straight line joining any two points in C is contained entirely in C. Thus define $H : X \times I \longrightarrow C$ by*

$$H(x, t) = (1 - t) \cdot f(x) + t \cdot g(x).$$

Then H is a homotopy from f to g. □

Example 6.3. *The n-disk D^n in $\mathbf{R}^n$ is a convex subset. Thus any two maps $f, g : X \longrightarrow D^n$ are homotopic. The same statement is far from being true (as we shall see later) if D^n is replaced by its boundary S^{n-1}. However, suppose that $f, g : X \longrightarrow S^{n-1}$ are two maps such that $f(x) \neq -g(x)$ for all $x \in X$, i.e. f and g never send a point of X into a pair of antipodal points on the sphere, or equivalently,* *the straight line segment between $f(x)$ and $g(x)$ never passes through the origin. Then the homotopy defined in the previous example can be divided by its norm to give a homotopy of f to g, i.e.*

$$H(x,t) = \frac{(1-t)\cdot f(x) + t\cdot g(x)}{||(1-t)\cdot f(x) + t\cdot g(x)||}$$

□

We are not quite equipped yet with tool appropriate for showing that two given maps are not homotopic to each other. However to motivate the discussion here is an example where this claim is quite obvious.

Example 6.4. *Let $Y = \mathbf{C} \setminus \{0\}$ be the punctured complex plane with . Let $f : S^1 \longrightarrow Y$ be the inclusion of the unit circle. Let $g : S^1 \longrightarrow Y$ be the unit circle shifted away far enough not to enclose the puncture at the origin, say $g(e^{2\pi it}) = 2 + e^{2\pi it}$. It is easy to see, at least intuitively that there is no homotopy of f to g. Indeed, in order to deform f into g it will be necessary to "tear" the circle, thus destroying continuity.*

If $f, g : X \longrightarrow Y$ are two homotopic maps then we write $f \sim g$.

Theorem 6.5. *Let X and Y be spaces. Then homotopy of maps defines an equivalence relation on the set of all maps from X to Y.*

Proof. We need to show that the relation $\sim$ is reflexive symmetric and transitive. Reflexivity is clear, namely $f \sim f$ by the homotopy $H(x,t) = f(x)$ for all $t \in I$. If $H : X \times I \longrightarrow Y$ is a homotopy of f to g then $H' : X \times I \longrightarrow Y$ defined by

$$H'(x,t) = H(x, 1-t)$$

is a homotopy of g to f proving symmetry of the relation. Finally if $f \sim g$ and $g \sim h$ through homotopies H_1 and H_2 respectively define $H : X \times I \longrightarrow Y$ by

$$H(x,t) = \begin{cases} H_1(x, 2t) & 0 \le t \le \frac{1}{2} \\ H_2(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Thus H is a homotopy of f to h, proving transitivity. □

Definition 6.6. *For spaces X and Y denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.*

Example 6.7. *Let C be a convex subset of $\mathbf{R}^n$ and let X be any space. Then $[X, C]$ is a single point set since any two maps are homotopic.* □

The notion of homotopy between two maps $f, g : X \longrightarrow Y$, or "absolute homotopy" can be generalized to the so called relative homotopy.

Definition 6.8. *A topological pair is an ordered pair of spaces (X, A) where A is a subspace of X. If (X, A) and (Y, B) are pairs then a map of pairs $f : (X, A) \to (Y, B)$ is a continuous function $f : X \to Y$, such that $f(A) \subseteq B$.*

The idea of a relative homotopy is best explained by considering homotopies in general as paths in the appropriate mapping spaces. We have already seen that an ordinary homotopy from f to g, where $f, g : X \longrightarrow Y$, is a path in the space $\mathrm{Map}(X, Y)$ starting at f and ending at g. In a similar way if (X, A) and (Y, B) are topological pairs one may consider the mapping space $\mathrm{Map}(X, A; Y, B)$ of all maps of pairs $(X, A) \longrightarrow (Y, B)$. This set is a subset of $\mathrm{Map}(X, Y)$ and thus has a natural topology induced from the topology of $\mathrm{Map}(X, Y)$. If $f, g : (X, A) \longrightarrow (Y, B)$ are maps of pairs, then a relative homotopy between them is a path in $\mathrm{Map}(X, A; Y, B)$ starting at f and ending at g. In more explicit terms one has the following

Definition 6.9. *Let $f, g : (X, A) \longrightarrow (Y, B)$ be maps of topological pairs. We say that f is homotopic to g as maps of pairs, or that f is homotopic to g relative to A if there exists a homotopy*

$$H : X \times I \longrightarrow Y$$

such that

1. *$H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ namely H is a homotopy from f to g in the usual sense and*
2. *$H(a, t) \in B$ for every $a \in A$ and $t \in I$.*

If f and g are maps of pairs as above and f is homotopic to g relative to A, we write $f \sim g \,\mathrm{rel}\, A$. One can observe easily that homotopy relative to A is also an equivalence relation. The proof is along the lines of the absolute case.

Exercise 6.10. *Show that the relation of relative homotopy is an equivalence relation on the set of maps from (X, A) to (Y, B).*

Understanding homotopy classes of maps between two given spaces is in some sense the most fundamental problem in homotopy theory. As one expects from a fundamental problem it is of course a very hard one in general. We shall see later a few menthods that may become helpful in understanding homotopy classes of maps between two spaces.

Homotopy Equivalence - Homotopy Type. We have now introduced an equivalence relation on the set of all continuous functions from a space X to another space Y, which allows us to think about two maps as the same if they can be continuously deformed into each other. This enables us to define a similar concept for topological spaces.

Definition 6.11. *Let X and Y be spaces. We say that X and Y are homotopy equivalent and write $X \simeq Y$ if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg \sim id_Y$ and $gf \sim id_X$.*

Intuitively, X and Y are homotopy equivalent if each can be deformed into the other in a continuous manner. The first observation one makes is the following.

Lemma 6.12. *Homotopy equivalence is an equivalence relation on spaces.*

Proof. The proof is left as an exercise for the reader. □

Homotopy equivalence of spaces also has a relative version.

Definition 6.13. *Two pairs (X, A) and (Y, B) are said to be homotopy equivalent if there are maps of pairs $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ such that $fg \sim id_Y \operatorname{rel} A$ and $gf \sim id_X \operatorname{rel} B$.*

We proceed with some easy examples.

Example 6.14. *Every convex subset of $\mathbf{R}^n$ is homotopy equivalent to a single point. To see this let B be a convex subset of $\mathbf{R}^n$. Let $b_0 \in B$ be any point. For an arbitrary point $b \in B$ the line segment $(1-t)b + tb_0$, $0 \leq t \leq 1$ is contained in B. Let $\mathtt{pt}$ denote the one point space. Let $f : \mathtt{pt} \to B$ denote the map taking the single point in pt to b_0 and let $g : B \to \mathtt{pt}$ denote the constant map. Then gf is the identity on $\mathtt{pt}$ (and thus obviously homotopic to the identity). Define a homotopy*

$$H : B \times I \to B$$

by $H(b, t) = (1 - t)b + tb_0$. Then H is continuous (convince yourself) and satisfies $H(b, 0) = b$ and $H(b, 1) = b_0$ and so H is a homotopy of fg to the identity map of B.

Definition 6.15. *A space X is said to be contractible if it is homotopy equivalent to a single point space.*

There are more contractible spaces than just convex subsets of $\mathbf{R}^n$. Even among subsets of Euclidean space there are many subspaces which are contractible without being convex, as one can imagine. Intuitively a subset of $\mathbf{R}^n$ is contractible if and only if it has no holes. Writing an explicit homotopy equivalence however may be rather difficult. The point of view given by the following lemma is sometimes useful.

Lemma 6.16. *A space X is contractible if and only if there exists a point $x_0 \in X$ such that the identity map on X is homotopic to the constant map which takes every point in X to x_0.*

Proof. Assume X is contractible. Then there is a point $x_0 \in X$ such that the composite

$$X \xrightarrow{K} \mathtt{pt} \xrightarrow{x_0} X$$

is homotopic to the identity on X.

Conversely, if for some point $x_0 \in X$, the composite above is homotopic to the idenity on X, then X is contractible since the composite the other way $K \circ x_0$ is the identity on the single point space. □

Example 6.17. *The solid torus and the cylinder are both homotopy equivalent to the circle S^1. To see this notice that the solid torus is just the Cartesian product $S^1 \times D^2$ and D^2 is contractible. The Cylinder is $S^1 \times I$ and I is contractible. Thus both assertions follow from the lemma below.* □

Lemma 6.18. *Let X be any space and let C be a contractible space. Then $X \times C$ is homotopy equivalent to X.*

Proof. Let $H : C \times I \to C$ be a homotopy of the identity on C to the map taking C to a chosen point $c_0 \in C$. Let $j : X \to X \times C$ be the map sending x to (x, c_0) and let $p : X \times C \to X$ be the projection to the first coordinate. Then $pj = id_X$ and $jp(x, c) = (x, c_0)$. Define

$$G : X \times C \times I \to X \times C$$

by $G(x, c, t) = (x, H(c, t))$. Then G is continuous as the composition of two continuous functions and satisfies $G(x, c, 0) = (x, H(c, 0)) = (x, c_0)$ and $G(x, c, 1) = (x, H(c, 1)) = (x, c)$. □

The relation of homotopy equivalence divides the class of all spaces into equivalence classes. For a given space X the equivalence class of X under homotopy equivalence is called the homotopy type of X. Thus two spaces are said to have the same homotopy type if they are homotopy equivalent. Homotopy type is a much weaker topological invariant than homeomorphism type, the equivalence class of a space with resepct to the relation X is equivalent to Y if and only if X and Y are homeomorphic. However it is an extremely useful generalisation. Namely, two spaces that are not of the same homotopy type can never have the same topological type. Thus for example, to show that all the surfaces constructed in the classification theorem are indeed distinct, it suffices to show that they are not of the same homotopy type.

The homotopy type of a space is easier to determine than its homeomorphism type. However even this weaker invariant is an extremely rich one and in general the task of distinguishing two homotopy types is hopeless. The study of homotopy types is the main theme of "homotopy theory". In the next sections we will familiarise ourselves with some of the main tools in the subject.

7. The fundamental group

Given two spaces X and Y let $[X, Y]$ denote the set of all homotopy classes of maps from X to Y. More generally if (X, A) and (Y, B) are two topological pairs, let $[X, A; Y, B]$ denote the set of all relative homotopy classes of maps from (X, A) to (Y, B). These sets are generally just discrete sets of points. However, under certain hypotheses they are endowed with extra structure. The fundamental group of a space X is a canonical example of this situation. Before we define this construction let us start with some generalities concerning homotopy classes of maps.

The first important feature of $[X, Y]$ is the fact that if $f : X \to X'$ and $g : Y \to Y'$ then there are induced maps

$$f^{\#} : [X', Y] \to [X, Y] \quad \text{and} \quad g_{\#} : [X, Y] \to [X, Y'],$$

defined as follows: Let $[\alpha] \in [X', Y]$ be the homotopy class of some map $\alpha : X' \to Y$. Define $f^{\#}[\alpha] = [\alpha \circ f]$. If $[\beta] \in [X, Y]$ is the homotopy class of some $\beta : X \to Y$, define $g_{\#}[\beta] = [g \circ \beta]$.

Lemma 7.1. *Let $f : Z \longrightarrow X$ and $g : Y \longrightarrow W$ be maps. Then the induced maps*

$$f^{\#} : [X, Y] \longrightarrow [Z, Y] \quad \text{and} \quad g_{\#} : [X, Y] \longrightarrow [X, W]$$

are well defined.

Proof. We prove the claim for $f^{\#}$ and leave the other case for the reader as the proof is analoguous. Thus we must show that if $\alpha \sim \alpha' : X \longrightarrow Y$ then $\alpha \circ f \sim \alpha' \circ f$. Indeed, let $H : X \times I \longrightarrow Y$ be a homotopy from α to α'. Define

$$H^f : Z \times I \longrightarrow Y$$

by $H^f(z, t) = H(f(z), t)$. Then H^f is obviously continuous and $H^f(z, 0) = \alpha(f(z))$ and $H^f(z, 1) = \alpha'(f(z))$ as required. □

The second and probably most important feature of sets of homotopy classes of maps is their homotopy invariance.

Theorem 7.2. *Let $f, g : X \to Y$ be homotopic maps. Then for any space Z the induced maps*

$$f^{\#} : [Y, Z] \to [X, Z] \quad \text{and} \quad g^{\#} : [Y, Z] \to [X, Z]$$

and

$$f_{\#} : [Z, X] \to [Z, Y] \quad \text{and} \quad g_{\#} : [Z, X] \to [Z, Y]$$

coincide.

Proof. Let $[h]$ denote the homotopy class of some map $h : Y \to Z$. Then the claim that $f^{\#}[h] = g^{\#}[h]$ is equivalent to saying that $hf \simeq hg$, since $f^{\#}[h] = [hf]$ and $g^{\#}[h] = [hg]$. Proving the second claim is similar and so we discuss only the first case and leave the second for the reader. Let $H : X \times I \longrightarrow Y$ be a homotopy from f to g. We construct a homotopy $H_h : X \times I \longrightarrow Z$ by

$$H_h(x, t) = h(H(x, t)).$$

Then H_h is obviously continuous and $H_h(x,0) = h(H(x,0)) = h(f(x))$ and $H_h(x,1) = h(H(x,1)) = h(g(x))$. □

Corollary 7.3. *If $f : X \to X'$ and $g : Y \to Y'$ are homotopy equivalences then both $f^\# : [X', Y] \to [X, Y]$ and $g_\# : [X, Y] \to [X, Y']$ are bijections of sets.*

Proof. Let $\tilde{f}$ be a homotopy inverse for f, namely $f\tilde{f} \simeq id_{X'}$ and $\tilde{f}f \simeq id_X$. Then $f^\#\tilde{f}^\# = id^\#_{X'}$ on $[X', Y]$ which is the identity map. Conversely $\tilde{f}^\# f^\# = id^\#_X$ on $[X, Y]$ which is again the identity map. Thus $f^\#$ and $\tilde{f}^\#$ are inverses for each other and so $f^\#$ is an isomorphism of sets. The second statement is similar. □

A pointed map from one pointed space to another is simply a map of pairs $f : (X, x_0) \longrightarrow (Y, y_0)$. Also we will sometimes say that X is a pointed space without using the pair notation (X, x_0).

Lemma 7.4. *Let $(X, x_0), (Y, y_0)$ and (Z, z_0) be pointed spaces. Then*

$$[X \vee Y, (x_0, y_0); Z, z_0] \cong [X, x_0; Z, z_0] \times [Y, y_0; Z, z_0]$$

and

$$[X, x_0; Y \times Z, (y_0, z_0)] \cong [X, x_0; Y, y_0] \times [X, x_0; Z, z_0]$$

Proof. Given a pointed map $f : X \vee Y \longrightarrow Z$, let f_X and f_Y denote the respective restrictions. Define

$$\phi : [X \vee Y, (x_0, y_0); Z, z_0] \longrightarrow [X, x_0; Z, z_0] \times [Y, y_0; Z, z_0]$$

by $\phi([f]) = ([f_X], [f_Y])$. Conversely, define

$$\psi : [X, x_0; Z, z_0] \times [Y, y_0; Z, z_0] \longrightarrow [X \vee Y, (x_0, y_0); Z, z_0]$$

by $\psi([f], [g]) = [f \vee g]$. It is easy to see that ϕ and ψ are mutual inverses provided they are well defined.

Thus if $f, h : X \vee Y \longrightarrow Z$ are pointed maps such that $[f] = [h]$ then we must show that $f_X \simeq h_X$ and $f_Y \simeq h_Y$. Let $H : (X \vee Y) \times I \longrightarrow Z$ be a homotopy of f to h. Define $H_X : X \times I \longrightarrow Z$ to be the restriction of H to the subspace $X \times I$ of $(X \vee Y) \times I$. Similarly define H_Y. It is easy to verify that H_X is a homotopy of f_X to h_X and H_Y is a homotopy of f_Y to h_Y.

Conversely let $f, g : X \longrightarrow Z$ and $s, t : Y \longrightarrow Z$ be pointed maps such that $[f] = [g]$ and $[s] = [t]$. We must show that $[f \vee s] = [g \vee t]$. Let H_1 be a homotopy of f to g and let H_2 be a homotopy of s to t. Define

$$H : (X \vee Y) \times I \longrightarrow Z$$

by

$$H(a,t) = \begin{cases} H_1(a,t) & \text{if} a \in X \\ H_2(a,t) & \text{if} a \in Y \end{cases}$$

Since X and Y intersect in the wedge only on their mutual base points, it is clear that H is well defined and continuous. It is now easy to verify that H is a homotopy as required □

We are now ready to define a sequence of extremely important topological invariants. Namely homotopy groups. However, in this course only one of those groups will be discussed in detail, namely the so called fundamental group.

Definition 7.5. *Let (X, x_0) be a pointed space. Consider the n-sphere S^n, $n \geq 0$ as a pointed space by taking the point $(1, 0, \cdots, 0) \in \mathbf{R}^n$ as a basepoint b_0. The n-th homotopy group $\pi_n(X, x_0)$ of (X, x_0) is defined to be the set of relative homotopy classes of maps $[S^n, b_0; X, x_0]$. In particular $\pi_1(X, x_0)$ is called the fundamental group of X at x_0.*

Exercise 7.6. *Show that the set $\pi_0(X, x_0)$ is the set of path components of X*

Let us now think about the fundametal group, in slightly intuitive terms. By definition the fundamental group of a pointed space (X, x_0) is the set of homotopy classes of maps from S^1 to X sending b_0 to x_0. Thus it is the set of homotopy classes of loops in X, which start and end at x_0. How can such a set become a group? If λ and ω are two loops in X starting and ending at x_0 one can imagine "adding them up" by defining a new loop, i.e. a map from S^1 to X, which goes half the time along λ and the second half along ω. This is an operation which takes two loops and adds them together thus defining a binary operation on the set of all loops in X which start and end at the base point. One could wonder whether this makes this set into a group.

Unfortunately, this is not quite the case. Namely the operation we have just defined is not associative and does not have a unit. Indeed, if you let λ, ω and ν be three loops, multiplying them as $(\lambda\omega)\nu$ means letting the product loop go the for first quarter along λ, for the second quarter along ω and for the last half of the time along ν. On the other hand the product $\lambda(\omega\nu)$ goes for the first half through λ and then the second half is split in two between ω and ν. These are simply not the same maps. Also the natural choice for a unit is the constant loop at the base point. But multiplying an arbitrary loop by the constant loop in any order does not give the original loop, but rather a new loop that goes through the original one twice as fast and then stays put for the other half.

It is easy to imagine though why multiplication in this set of loops is associative and has a unit up to homotopy. That is to say, although the two ways of multiplying loops $(\lambda\omega)\nu$ and $\lambda(\omega\nu)$ do not give the same loop do give homotopic loops. Multiplying with the constant loop does not yield the original loop but does give one homotopic to it. Thus if instead of considering the loops themselves we consider homotopy classes of loops, we will get a group. This is the idea behind the definition of the fundamental group, originally due to Poincaré. We shall now proceed by making the discussion more rigorous.

We now explicitly define multiplication in the fundamental group and prove its basic properties.

Definition of multiplication in $\pi_1(X, x_0)$. Consider the circle S^1. There is a map called the "pinch map"

$$p : S^1 \longrightarrow S^1 \vee S^1$$

given by identifying the base point with its antipodal. Notice that p is a based map, namely it sends the base point of S^1 to the base point of the wedge. If one considers S^1 as the set of all complex numbers of the form $e^{2\pi it}$ for $t \in I$, then the pinch map is defined by

$$p(e^{2\pi it}) = \begin{cases} e^{2\pi i\cdot 2t} & \text{if } 0 \le t \le \frac{1}{2} \\ e^{2\pi i\cdot(1-2t)} & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

On the other hand for every pointed space X there is a canonical map

$$\phi : X \vee X \longrightarrow X$$

called the folding map, given by sending each factor to X by the identity map.

By Lemma 7.4 for any pointed space (X, x_0) there is an isomorphism of sets

$$[S^1 \vee S^1, b_0; X, x_0] \cong \pi_1(X, x_0) \times \pi_1(X, x_0).$$

The operation on $\pi_1(X, x_0)$ is by definition, the map induced by p. More explicitely, given two homotopy classes $[f], [g] \in \pi_1(X, x_0)$ represented by pointed loops $f, g : S^1 \longrightarrow X$, their product $[f][g]$ is the homotopy class of the loop given by the composition

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{f\vee g} X \vee X \xrightarrow{\phi} X.$$

Associativity. The next lemma is the major step in proving associativity of this multiplication.

Lemma 7.7. *The maps*

$$l : S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{p\vee id} (S^1 \vee S^1) \vee S^1$$

and

$$r : S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{id\vee p} S^1 \vee (S^1 \vee S^1)$$

are homotopic.

Proof. Write

$$l(e^{2\pi it}) = \begin{cases} (e^{2\pi i4t})_1 & 0 \le t \le \frac{1}{4} \\ (e^{2\pi i(4t-1)})_2 & \frac{1}{4} \le t \le \frac{1}{2} \\ (e^{2\pi i(2t-1)})_3 & \frac{1}{2} \le t \le 1 \end{cases} \quad \text{and} \quad r(e^{2\pi it}) = \begin{cases} (e^{2\pi i2t})_1 & 0 \le t \le \frac{1}{2} \\ (e^{2\pi i(4t-2)})_2 & \frac{1}{2} \le t \le \frac{3}{4} \\ (e^{2\pi i(4t-3)})_3 & \frac{3}{4} \le t \le 1 \end{cases}$$

Consider the map $f : I \longrightarrow I$ given by

$$f(t) = \begin{cases} 2t & 0 \le t \le \frac{1}{4} \\ t + \frac{1}{4} & \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t+1}{2} & \frac{1}{2} \le t \le 1 \end{cases}$$

Since I is a convex subset of $\mathbb{R}$ the map f is homotpic to the identity on I. Furthermore, notice that f sends the boundary points 0 and 1 in I to themselves. The map

f is homotopic to the identity relatively to the boundary $\partial I \subset I$. Indeed, there is a straight line homotopy between f and the identity, given by

$$H(t,s) = (1-s)t + sf(t) = \begin{cases} (1-s)t + 2st & 0 \leq t \leq \frac{1}{4} \\ (1-s)t + s(t+\frac{1}{4}) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ (1-s)t + s\frac{t+1}{2} & \frac{1}{2} \leq t \leq 1 \end{cases}$$

which is easily seen to preserve the boundary, namely $H(0,s) = 0$ and $H(1,s) = 1$ for all $s \in I$. Define $G : S^1 \times I \longrightarrow S^1 \vee S^1 \vee S^1$ by $G(e^{2\pi it}, s) = r(e^{2\pi i H(t,s)})$. Then G is easily checked to be continuous and $G(t,0) = r(e^{2\pi i H(t,0)}) = r(e^{2\pi it})$ and

$$G(t,1) = \begin{cases} e^{2\pi i f(t)} & 0 \leq f(t) \leq \frac{1}{2} \\ e^{2\pi i(4f(t)-2)} & \frac{1}{2} \leq f(t) \leq \frac{3}{4} \\ e^{2\pi i(4f(t)-3)} & \frac{3}{4} \leq f(t) \leq 1 \end{cases} = \begin{cases} e^{2\pi i 2t} & 0 \leq t \leq \frac{1}{4} \\ e^{2\pi i(4t-1)} & \frac{1}{4} \leq t \leq \frac{1}{2} \\ e^{2\pi i(2t-1)} & \frac{1}{2} \leq t \leq 1 \end{cases} = l(e^{2\pi it}).$$

□

Notice also that the homotopy H constructed above is pointed, namely for all $s \in I$ one has $G(1,s) = r(1) = 1$. Now that we have a homotopy between the maps r and l we are ready to prove associativity of multiplication in the fundamental group. Indeed by Lemma 4.18 one has

$$[S^1 \vee S^1 \vee S^1, b_0; X, x_0] \cong \pi_1(X,x_0) \times \pi_1(X,x_0) \times \pi_1(X,x_0).$$

Thus the maps r and l induce two functions

$$r^*, l^* : \pi_1(X,x_0) \times \pi_1(X,x_0) \times \pi_1(X,x_0) \longrightarrow \pi_1(X,x_0).$$

If $[\alpha], [\beta], [\gamma] \in \pi_1(X,x_0)$ then by definition

$$r^*([\alpha],[\beta],[\gamma]) = [\alpha]([\beta][\gamma]) \quad \text{and} \quad r^*([\alpha],[\beta],[\gamma]) = ([\alpha][\beta])([\gamma]).$$

But we have shown that r and l are homotopic and therefore the induced maps coincide, proving that multiplication on $\pi_1(X,x_0)$ is an associative operation.

Before we proceed, let us recapitulate on what was done here. We defined the multiplication on $\pi_1(X,x_0)$ as the map induced by the pinch map. Then we observed that the two ways of pinching a circle into a wedge of three circles are homotpic. Notice that the proof of this fact depends on the fact that the map f defined in the proof is homotopic to the identity map on I relative to ∂I. This map simply reparametrises the partition for the interval which is used to define r into the partition used to define l. Since this gives the two possible ways of multiplying three elements in $\pi_1(X,x_0)$, we obtained associativity of multiplication. The proofs of existence of a unit and inverses for the multiplication can be done in a very similar fashion. We now prove the existence of a unit following the same lines. However, this time instead of constructing an explicit homotopy we use the following

Lemma 7.8. *Let $f : I \longrightarrow I$ be any self map of the unit interval which preserves the boundary ∂I, namely $f(0) = 0$ and $f(1) = 1$. Then f is homotopic to the identity relative to ∂I.*

Proof. Define a straight line homotopy $F : I \times I \longrightarrow I$ by

$$F(t,s) = (1-s)t + sf(t).$$

Then $F(t,0) = t$, $F(t,1) = f(t)$, $F(0,s) = 0$ and $F(1,s) = f(1) = 1$. $\square$

Existence of an Identity Element. Let 1 denote the class of the constant map in $\pi_1(X, x_0)$. We must show that for any pointed loop α in X, $1[\alpha] = [\alpha]1 = [\alpha]$. Again consider the reparametrisation of the interval given by

$$f(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ 2t-1 & \frac{1}{2} \le t \le 1 \end{cases}$$

Notice that, as before, f preserves the boundary ∂I. Given a based loop $\alpha : S^1 \longrightarrow X$ the element $1[\alpha]$ in $\pi_1(X, x_0)$ is given by the class of the composition $\alpha(e^{2\pi i f(t)})$. But by Lemma 7.8, $f(t)$ as a self map of the unit interval is homotopic to the identity map relative to the boundary ∂I. And so $\alpha(e^{2\pi i f(t)})$ is homotopic to $\alpha(e^{2\pi i t})$ as a pointed map, showing that $1[\alpha] = [\alpha]$. The proof that 1 is a right unit is similar and is left for the reader.

Existence of inverses. For the proof of existence of inverses we use a little trick. First we need a lemma, which is useful in many contexts.

Lemma 7.9. *Let $f : X \longrightarrow Y$ be a null-homotopic map. Then for any two maps $Y \xrightarrow{g} Z$ and $W \xrightarrow{h} X$, the composites $g \circ f$ and $f \circ h$ are also null homotopic.*

Proof. We prove the claim for the composite $g \circ f$. The other composite is null-homotopic by analogy. Let $H : X \times I \longrightarrow Y$ be a homotopy of f to a constant map. Then The composite

$$X \times I \xrightarrow{H} Y \xrightarrow{g} Z$$

gives a homotopy of $g \circ f$ to a constant map. $\square$

For any based loop α representing $[\alpha]$ in $\pi_1(X, x_0)$ let $[\alpha]^{-1}$ be the class represented by the loop $\alpha^{-1}(e^{2\pi i t}) = \alpha(e^{2\pi i(1-t)})$. Then by definition $[\alpha][\alpha]^{-1}$ is represented by

$$\alpha * \alpha^{-1}(e^{2\pi i t}) = \begin{cases} \alpha(e^{2\pi i 2t}) & 0 \le t \le \frac{1}{2} \\ \alpha^{-1}(e^{2\pi i(2t-1)}) & \frac{1}{2} \le t \le 1 \end{cases} = \begin{cases} \alpha(e^{2\pi i 2t}) & 0 \le t \le \frac{1}{2} \\ \alpha(e^{2\pi i(2-2t)}) & \frac{1}{2} \le t \le 1 \end{cases}$$

Equivalently $\alpha * \alpha^{-1}$ can be written the composite

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{1\vee -1} S^1 \vee S^1 \xrightarrow{\alpha\vee\alpha} X \vee X \xrightarrow{\phi} X.$$

The reader should have no trouble verifying that the composite $\phi \circ (\alpha \vee \alpha)$ is identically the same as the composition

$$S^1 \vee S^1 \xrightarrow{\phi} S^1 \xrightarrow{\alpha} X.$$

Consequently $\alpha * \alpha^{-1}$ can be written as the composition

$$\alpha\phi(1 \vee -1)p$$

and by Lemmma 7.9 it suffices to show that $\phi(1 \vee -1)p$ is null-homotopic. Notice that this composite is simply the map which wraps the circle around itself forth and back. Thus it is intuitively clear why it is null-homotopic. However, for the sake of rigor, we give a proof of this statement.

Lemma 7.10. *The map*

$$\phi(1 \vee -1)p : S^1 \longrightarrow S^1$$

is null-homotopic relative to the base point.

Proof. Let $\xi : S^1 \longrightarrow I$ be the map defined by

$$\xi(e^{2\pi it}) = \begin{cases} 2t & 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then one easily verifies that $exp \circ \xi = \phi(1 \vee -1)p$. Hence we have factored the right hand side through I, which is a contractible space. It follows that $\phi(1 \vee -1)p$ is null-homotopic. Furthermore, one can always construct a homotopy $H : S^1 \times I \longrightarrow I$ of ξ to the constant map sending S^1 to 0 with the additional property that $H(1,t) = 0$. Define $G : S^1 \times I \longrightarrow S^1$ by

$$G(b,t) = (exp(H(b,t)).$$

Then $G(b,0) = exp(H(b,0)) = exp(\xi(b)) = \phi(1 \vee -1)p(b)$, $G(b,1) = exp(H(b,1)) = exp(0) = 1$ and $G(1,t) = exp(H(1,t)) = exp(0) = 1$. This completes the proof. □

The lemma now implies by the discussion above, the existence of invereses in $\pi_1(X, x_0)$.

To summarise our observations we consider once more what has been done. Multiplication on the fundamental group was defined as the map induced by the pinch map on S^1. Namely $p : S^1 \longrightarrow S^1 \vee S^1$ induces a map

$$p^{\#} : [S^1 \vee S^1, b_0; X, x_0] \longrightarrow [S^1, b_0; X, x_0].$$

The right hand side is by definition the fundamental group $\pi_1(X, x_0)$ and by Lemma 7.4 the left hand side is $\pi_1(X, x_0) \times \pi_1(X, x_0)$. We than used only the features of the pinch map to conclude associativity, the existence of a unit and the existence of inverses. Indeed the fact that the fundamental group of a space X is indeed a group depends entirely on the fact that S^1 admits a pinch map. An important side remark at this point is that every suspension space admits a pinch map, and indeed if X is a suspension space and Y is arbitrary, the existence of the pinch map implied that the set of pointed homotopy classes of maps from X to Y forms a group under multiplication induced by the pinch map.

Our observations have thus produced a group associated to each pointed space (X, x_0). If $f : X \longrightarrow Y$ is a pointed map then one obtains an induced map

$$f_{\#} : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

The next question one may naturally ask is whether or not this map is a homomorphism.

The approach we took makes it very easy to observe that the answer to this question is positive. Indeed let $[\alpha], [\beta] \in \pi_1(X, x_0)$ be represented by maps α and β respectively. Then $[\alpha][\beta]$ is represented by the class of the composition

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\phi} X.$$

Then $f_\#([\alpha][\beta])$ is represented by $f \circ \phi \circ (\alpha \vee \beta) \circ p$. But $f \circ \phi$ is identically the same as the composite

$$X \vee X \xrightarrow{f\vee f} Y \vee Y \xrightarrow{\phi} Y.$$

Composing the last with $(\alpha \vee \beta) \circ p$ one obtains a map representing $f_\#[\alpha]f_\#[\beta]$. And so we conclude that $f_\#$ is a homomorphism of groups. Notice that we did not check that the identity element of $\pi_1(X, x_0)$ is sent to the identity of $\pi_1(Y, y_0)$ under $f_\#$, but this is obvious. We summarise our results in the following

Theorem 7.11. *Let X be a pointed space. Then the set $[S^1, b_0; X, x_0]$ of pointed homotopy classes of maps forms a group under the map induced by the pinch map on S^1. Furthermore, if $f : X \longrightarrow Y$ is a pointed map then the induced map*

$$f_\# : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is a group homomorphism.

Another question one may wonder about is the dependence of $\pi_1(X, x_0)$ on the choice of the base point x_0.

Exercise 7.12. *Show that if X is path-connected then $\pi_1(X, x_0)$ does not depend of the choice of a base point.*

Comments on Calculations. Calculation of the fundamental group of a space is generally not easy. One can show that contractible spaces as well as spheres of dimension at least 2 have a trivial fundamental group. Spaces with a trivial fundamental group are called simply-connected or 1-connected. One can also calculate the fundamental group of a product and of a wedge in terms of the fundamental groups of the factors. The first non-trivial example of a non-vanishing fundamental group is given by $\pi_1(S^1)$. However in order to be able to give a rigorous calculation of it we shall need the machinery of "covering spaces" which will be covered in the next section. This will also enable us to compute the fundamental group of many other spaces. We conclude this section by an intuitive discussion of the fundamental group of some simple spaces.

Consider the circle S^1 first. Based loops in S^1 are in some sense determined up to homotopy by the number of times the loop completes an entire circle and in what orientation. There are canonical choices for loops which go around the circle exactly n times for any integer n, namely the loops taking z to z^n. These loops turn out to represent all the elements in $\pi_1(S^1)$. Thus $\pi_1(S^1) \cong \mathbf{Z}$. As we shall see later the fact that S^1 is the orbit space of a $\mathbf{R}$ under the natural $\mathbf{Z}$ action is closely related to the fundamental group of the circle being the integers.

Exercise 7.13. *Calculate the fundamental group of the n-fold torus* $T^n = S^1 \times \cdots S^1$*, n-times.*

Next consider the real projective space $\mathbf{R}P^2$. This space can be considered as a hemisphere with an identification on its boundary, which gives the prjective plane. The identification indicates that one loop around the boundary is not null-homotopic. However if one continues to loop around once more then the resulting loop is null-homotopic. The outcome is that the fundamental group of $\mathbf{R}P^2$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

The fundamental group is an extremely important topological invariant. We now assume that the calculations we intuitively carried out above are indeed correct and use them to prove geometric results.

Brouwer's Fixed Point Theorem. The Brouwer fixed point theorem is the statement that a self map of a convex subset of $\mathbf{R}^n$ always has a fixed point. We shall prove this for the disk.

Theorem 7.14. *Let* D^2 *denote the closed disk. Then any self map* $f : D^2 \longrightarrow D^2$ *has a fixed point, namely, there is some* $x \in D^2$ *such that* $f(x) = x$*.*

Proof. Assume the theorem is false. Thus there exists a self map f of D^2 which has no fixed point. Define a new function $g : D^2 \longrightarrow \partial D^2 = S^1$ as follows. For every $x \in D^2$ consider the line segment from $f(x)$ to x and let it continue until it intersects with the boundary S^1 at some point y. Define $g(x) = y$. Then continuity of f guarantees that g is continuous as well and notice that if $x \in \partial D^2$ then by construction $g(x) = x$. Now there is an inclusion of S^1 in D^2 as the boundary and so there is a factorisation of the identity map on S^1 as follows

$$S^1 \stackrel{inc}{\longrightarrow} D^2 \stackrel{g}{\longrightarrow} S^1.$$

But applying the fundamental group operator this gives a sequence of groups and homomorphisms

$$\mathbf{Z} \longrightarrow \{1\} \longrightarrow \mathbf{Z}.$$

The composition must be the identity being induced by the identity map, but the identity on the integer cannot be factored through the trivial group and thus we obtain a contradiction completing the proof of the theorem. □

The Degree of a self map of S^1. Consider $\pi_1(S^1) \cong \mathbf{Z}$. Take the identity map as a representative of the class corresponding to 1 under the above isomorphism. The integer 0 corresponds obviously to the class of the constant map in $\pi_1(S^1)$. More generally, let $f : S^1 \longrightarrow S^1$ be any map. Then f represents some homotopy class in $\pi_1(S^1)$. Let $deg(f)$ denote the integer corresponding to the homotopy class of f.

Corollary 7.15. *A self map of* S^1 *is determined up to homotopy by its degree. In particular a self map of* S^1 *is null homotopic if and only if it has degree 0.*

Obviously, all this depends on proving first that $\pi_1(S^1)$ is indeed isomorphic to the additive group of the integers. This can be done by defining the degree geometrically and showing that a sef map of S^1 is determined up to homotopy by its degree, which is more or less what we will do later on.

One observation important for the sequel is that the degree of a map behaves as one expects from the special maps sending z to z^n, i.e. if fg denotes the self map of S^1 obtained by poinwise multiplication of f and g then $deg(fg) = deg(f) + deg(g)$ whereas composition gives $deg(f \circ g) = deg(f) \cdot deg(g)$.

The Fundamental Theorem of Algebra. We now present one of the most classical theorems in the history of mathematics, which admits several proofs, some of which the reader might already be familiar with.

Theorem 7.16 (Fundamental Theorem of Algebra)**.** *Every polynomial with complex coefficients has a root.*

Proof. We shall assume the converse and find a contradiction. Let $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$. We may assume without loss of generality that $a_n = 1$. Define a map

$$\phi : S^1 \times \mathbf{R}_{\geq 0} \longrightarrow S^1$$

by $G(z,r) = \frac{P(rz)}{|P(rz)|}$. Since $P(z)$ is assumed to have no roots the map G is well defined and continuous. For any non-negative real number r, define $g_r : S^1 \longrightarrow S^1$ by $g_r(z) = G(z,r)$. We first claim that all g_r are homotopic.

Indeed, let r, s be non-negative real numbers. Define

$$H_{r,s} : S^1 \times I \longrightarrow S^1$$

by $H_{r,s}(z,t) = G(z,(1-t)r + ts)$. Then $H_{r,s}$ is continous and $H_{r,s}(z,0) = G(z,r) = g_r(z)$ and $H_{r,s}(z,1) = G(z,s) = g_s(z)$, proving the claim. Notice that $g_0(z) = G(z,0)$ is a constant map and thus is null-homotopic so has degree 0.

For a positive real number M consider the rational function $T(M,z) = \frac{P(Mz)}{(Mz)^n}$. Then one has

$$\lim_{M\to\infty} T(M,z) = 1$$

for every z. If the norm of z is bounded then one can choose a real number M sufficiently large such that the function $T(M,z)$, considered as a function of z has a positive real part for all z os lesser norm. In particular such an M can be chosen so that this holds for all $z \in S^1$.

For such an M we have

$$T(M,z)\left|\frac{(Mz)^n}{P(Mz)}\right| = \frac{g_M(z)}{z^n}$$

also has a positive real part for every $z \in S^1$. Consequently the map sending z to $\frac{g_M(z)}{z^n}$ is not onto and hence has degree 0. On the other hand the degree of this map is equal to the difference between the degree of $g_M(z)$ and the degree of the map sending z to z^n i.e. n. This implies that the degree of g_M is equal to n and hence a contradiction. The theorem follows. □

8. Calculation of the Fundamental Group

This section is devoted entirely to the fundamental group and in particular to some elementary calculations and further applications.

The two main computational tools in calculating fundamental groups are the theory of covering spaces and the Van Kampen theorem. A deep discussion of covering spaces is beyond the scope of this course. Instead we shall start with a special case already mentioned in the previous section, namely the circle. Then we will state the Van Kampen and give a sketch of its proof. This collection of tools will enable us to carry out a good number of calculations. In particular we will be able to distinguish the homotopy types of the various closed surfaces given in the classification theorem.

8.1. **The Circle.** The study of the circle presented here is in fact an discussion of covering spaces in disguise. However as this is the simplest example of a covering space it is likely to be instructive.

For every integer n let $\gamma_n : I \longrightarrow S^1$ denote the map $\gamma_n(t) = e^{2\pi int}$. Notice that each γ_n is a pointed loop in S^1 if the point $1 \in S^1$ is taken as the base point. There is a function

$$\psi : \mathbf{Z} \longrightarrow \pi_1(S^1)$$

given by $\phi(n) = [\gamma_n]$.

Lemma 8.1. *The map ψ defined above is a homomorphism of groups.*

Proof. Obviously 0 goes to the constant map. Thus we need to show that $[\gamma_n][\gamma_k] = [\gamma_{n+k}]$. But $[\gamma_n][\gamma_k]$ is the class of the composition

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{\gamma_n \vee \gamma_k} S^1 \vee S^1 \xrightarrow{\phi} S^1.$$

By inspection this map "goes around" the target circle $n + k$ times and differs from γ_{n+k} only by parametrisation. Hence the two are homotopic and the lemma follows. □

Our task will be to show that ψ is an isomorphism. We start by arguing that it is onto. Consider the exponential map $e : \mathbf{R} \longrightarrow S^1$ given by $e(r) = e^{2\pi ir}$. We will show that every α from the interval to S^1 can be "lifted" to a path in $\mathbf{R}$ starting at 0, i.e. there exists a map $\gamma : I \longrightarrow \mathbf{R}$ such that $\gamma(0) = 0$ and $\alpha = e\gamma$. Then if we start with a loop α in S^1 then the end point of a lift γ is at some integer n. This integer is by definition the degree of α. Furthermore, the map γ_n defined above is just multplication by n from I to $\mathbf{R}$ followed by the exponential map. Since any two maps from I to $\mathbf{R}$ with the same end points are homotopic rel∂I, we obtain that $< \alpha >= \psi(n) =< \gamma_n >$.

Lemma 8.2 (Unique path lifting property)**.** *Let $\alpha : I \longrightarrow S^1$ be any loop based at 1. Then there exists a unique lift $\tilde{\alpha} : I \longrightarrow \mathbf{R}$ such that $\tilde{\alpha}(0) = 0$ and $e\tilde{\alpha} = \alpha$.*

Proof. Consider the open subsets $U = S^1 \setminus \{1\}$ and $V = S^1 \setminus \{-1\}$. Then $U \cup V$ is an open covering of S^1. Furthermore, $e^{-1}(U)$ is a disjoint union of all oepn intervals $(n, n + 1)$ in $\mathbf{R}$. Each one of these intervals is mapped homeomorphically into U by the exponential map. Similarly $e^{-1}(V)$ is a disjoint union of all open intrervals of the form $(n-\frac{1}{2}, n+\frac{1}{2})$ and each of these is mapped homeomorphically onto V by e.

Let α be a loop in S^1. By Lebesgue's lemma there is a partition of the interval $0 = t_0 < t_1 < \cdots < t_k = 1$, such that the image of α restricted to $[t_i, t_{i+1}]$ is contained entirely in either U or V. Consider the first sub interval $[0, t_1]$. Then the image of α on it is contained in V because it certainly contains the point 1. Let $\sigma_1 : V \longrightarrow \mathbf{R}$ be the inverese of e restricted to $(-\frac{1}{2}, \frac{1}{2})$. Then $\tilde{\alpha}_1 = \sigma_1\alpha$ is a lift of the restriction of α to $[0, t_1]$.

Assume by induction that $\tilde{\alpha}_k$ has been defined lifting the restriction of α to $[0, t_k]$ into $\mathbf{R}$. The image of $[t_k, t_{k+1}]$ under α is contained either in U or in V. Suppose that $\tilde{\alpha}_k(t_k) = r_k$. Let W denote either U or V so that $\alpha([t_k, t_{k+1}] \subseteq W$. Let $\sigma_{k+1} : W \longrightarrow \mathbf{R}$ be the inverse of e restricted to the component of $e^{-1}(W)$ containing r_k. Then $\sigma_{k+1}\alpha$ is a list of the restriction of α to $[t_k, t_{k+1}]$ which agrees with $\tilde{\alpha}_k$ on t_k. Thus define $\tilde{\alpha}_{k+1}$ to be the extension of $\tilde{\alpha}_k$ to $[0, t_{k+1}]$ obtained by glueing together $\tilde{\alpha}_k$ with $\sigma_{k+1}\alpha$. This completes the inductive step.

Notice that at every stage there was a unique choice of a lift due to the maps σ_k being inverses of a given homeomorphism. Thus the lift we obtained is unique. □

In order to prove that ψ is 1-1, we need to have a homotopy lifting lemma, which can be used to lift a homotopy from the circle to the real line.

Lemma 8.3 (Homotopy lifting property)**.** *Let $F : I \times I \longrightarrow S^1$ be a map such that $F(0, t) = F(1, t) = 1$ for all t. Then there exists a unique homotopy $\tilde{F} : I \times I \longrightarrow \mathbf{R}$ such that $e\tilde{F} = F$ and $\tilde{F}(0, t) = 0$ for all t.*

Proof. The idea is more or less the same as the proof of unique path lifting. Namely, Lebesgue's lemma applies to give a partition of the square $I \times I$ into sub squares $S_{i,j} = [t_i, t_{i+1}] \times [s_j, s_{j+1}]$, such that F sends $S_{i,j}$ into either U or V as above. A lift is then constructed along the same lines. Details are left for the reader. □

We are now ready to prove that ψ is 1-1. Suppose $\phi(n) = 1$ in $\pi_1(S^1)$. Thus there is a path γ in $\mathbf{R}$ connecting 0 to n such that $\pi\gamma$ is a null-homotopic loop in S^1. Let $F : I \times I \longrightarrow S^1$ be a specific null homotopy of the constant loop at 1 to $\pi\gamma$. By the homotopy lifting property, there is a lift $\tilde{F} : I \times I \longrightarrow \mathbf{R}$ such that $\pi\tilde{F} = F$ and $\tilde{F}(0, t) = 0$ for all t.

The homotopy F sends the subset of $I \times I$ given by the left right and bottom sides of the square to 1. Since $\tilde{F}$ is a lift and $\tilde{F}(0, t) = 0$ it follows that the same subset is being sent to 0 under $\tilde{F}$. Considet the path in $\mathbf{R}$ given by $\tilde{F}(s, 1)$. It is a lift of $\pi\gamma$, which starts at 0. By unique path lifting $\tilde{F}(s, 1) = \gamma(s)$ for all s. But $\tilde{F}(1, 1) = \gamma(1) = 0$ and so $n = 0$ and ψ is 1-1.

8.2. **Orbit Spaces.** The circle S^1 is an orbit space of $\mathbf{R}$ by the action of the integer. The fact that $\pi_1(S^1)$ truns out to be the integers is not a coincidence. We now conisder a much larger family of such cases.

Definition 8.4. *Let G be a discrete group and ler X be a G space. We say that X is a free G space if for every $x \in X$ there is a neibourhood U_x of X such that for every non-trivial element $g \in G$, $U_x \cap gU_x = \emptyset$.*

For instance the antipodal action of $\mathbf{Z}/2\mathbf{Z}$ on the sphere S^n is a free action (check). The following theorem is very useful for calculation of the fundamental group. Its proof is similar to the calculation of $\pi_1(S^1)$, but there are some additional technical difficulties due to its generality and will thus be omitted at this stage.

Theorem 8.5. *Let G be a discrete group and let X be a simply-connected free G space. Then the orbit space X/G is path-connected and $\pi_1(X/G) \cong G$.*

Proof. We give only a sketch of the proof. If G acts freely on X, then for every point $y \in X/G$ there exists an open neighbourhood $U_x \subseteq X/G$ of x such that $\pi^{-1}(U_x)$ is a disjoint union of open subsets of X, one for every element of G, each of which is mapped homeomorphically onto U_x via the projection π. This is sufficient to prove that $\pi : X \longrightarrow X/G$ has both the unique path lifting property and the homotopy lifting property. The proof goes along the same line as it was done for the circle.

One now needs to construct a homomorphism of G into $\pi_1(X/G)$. Let $a \in \pi_1(X/G)$ be any element. Then a is represented by some based loop α in X/G. Let $\bar{x}_0$ denote the base point in X/G. Let $x \in \pi^{-1}(x_0)$ be any point. Then unique path lifting gives a unique path ω_x starting at x and projecting down to α via π. Since the composition $\pi\omega_x$ is a loop in X/G, $\omega_x(1) \in \pi^{-1}(\bar{x}_0)$. Thus there exists a unique element $g \in G$ such that $gx = \omega_x(1)$. Define $\psi : \pi_1(X/G) \longrightarrow G$ by $\psi(a) = g$. The proof that ψ is a well defined homomorphism is quite straight forward. The proof that it is an isomorphism is a bit more involved but similar to the special case of the circle and the real line and we omit it. □

Corollary 8.6. *For every $n > 1$, $\pi_1(\mathbf{R}P^n) \cong \mathbf{Z}/2\mathbf{Z}$.*

Proof. The real projetive space $\mathbf{R}P^n$ is the orbit space of a free $\mathbf{Z}/2\mathbf{Z}$ action on the sphere, which for $n > 1$ is simply-connected. □

Further examples of calculations of the same kind will appear in the coming tutorial sheet. We conclude the discussion of covering spaces with two more sample calculations, one of which you should already be familiar with.

Consider first the torus T^2. Recall that it can be expressed as the product $S^1 \times S^1$. In a recent homework assignment you were required to show that the fundamental group of a product space is isomorphic to the product of fundamental groups of the factors. We have shown that $\pi_1(S^1) \cong \mathbf{Z}$. It thus follows that

$$\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) = \mathbf{Z} \times \mathbf{Z}.$$

We now present a different proof of the same result. Namely, recall that the torus can be obtained as the orbit space of the the real plane $\mathbf{R}^2$ by the action of the group $\mathbf{Z} \times \mathbf{Z}$. The action is given by

$$(m, n) \cdot (x, y) = (x + m, x + n).$$

It is easy to verify that this action is free (convince yourself). Thus we obtain the same result as above using Theorem 5.5 because the plane $\mathbf{R}^2$ is simply-connected (in fact even contractible).

Next we consider the Klein bottle K^2. It turns out that it can also be obtained as an orbit space of the plane $\mathbf{R}^2$ but with respect to an action of a different group. To present calculation some definitions are required.

Definition 8.7. *A free group on a set $\{x_1, x_2, \cdots, x_n\}$ is the group of all words in the x_i and their inverses, with product given by juxtaposition and the only cancelations allowed are the obvious ones. If the set of generators is denoted by X then the resulting free group is denoted by $F(X)$.*

Definition 8.8. *A group G is said to be generated by a set $X = \{x_1, x_2, \cdots, x_n\}$ subject to the relations $R = \{r_1, r_2, \cdots, r_k\}$ if each r_j is a word in the free group $F(X)$ and G is the quotien group of $F(X)$ by its minimal normal subgroup containing the set R. In that case we write*

$$G = \langle X \mid R\rangle = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_k\rangle.$$

Example 8.9. *The cyclic group $\mathbf{Z}/n\mathbf{Z}$ can be written as $\langle x \mid x^n\rangle$. A direct product of two cyclic groups $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ can be written as*

$$\langle x, y \mid x^n, y^m, x^{-1}y^{-1}xy\rangle.$$

The free product $\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ of $\mathbf{Z}$ with $\mathbf{Z}/2\mathbf{Z}$ can be written as $\langle x, y \mid y^2\rangle$.

We can now discuss the Klein bottle. Consider the group

$$\pi = \langle s, t \mid s^{-1}tst\rangle.$$

The group π acts on the plane $\mathbf{R}^2$ as follows

$$t(x, y) = (x + 1, y) \quad \text{and} \quad s(x, y) = (1 - x, 1 + y).$$

One needs to verify that this is indeed a group action. In the context of generators and relations this amounts to checking that the relation is satisfied. Namely that $ts(x, y) = st^{-1}(x, y)$ for all $(x, y) \in \mathbf{R}^2$. But one has

$$ts(x, y) = t(1 - x, 1 + y) = (2 - x, 1 + y) \quad \text{and}$$
$$st^{-1}(x, y) = s(x - 1, y) = (1 - (x - 1), 1 + y) = (2 - x, 1 + y)$$

which proves the claim.

Inspection of the geometric effect of the π-action on $\mathbf{R}^2$ shows at once that

1. The action is free and
2. the orbit space is the usual identification space defining K^2.

Thus we obtain that $\pi_1(K^2) \cong \pi$.

We have seen above that if a space is an orbit space of a discrete group action on a simply-connected space, then the fundamental group of the orbit space is given by the operating group. The method is in fact more general and allows the calculation of the fundamental group of an orbit space even when the space operated upon is not simply-connected. However this requires a more thorough discussion of covering spaces, which we shall not attempt.

8.3. **The Van Kampen Theorem.** We now discuss another way of calculating the fundamental group of a space. This corresponds to the second general family of identification spaces we discussed, namely spaces which are obtained by glueing together two spaces along a common subspace. Before we can state the theorem we need some group theoretic preliminaries again.

Given two groups G and H containing a common subgroup K, one may wonder whether it is possible to "glue" G and H together along K. To make this precise, assume that there are two monomorphisms $f : K \longrightarrow G$ and $g : K \longrightarrow H$. We want to construst a new group P and maps $p : G \longrightarrow P$ and $q : H \longrightarrow P$, such that whenever L is an arbitrary group and $\alpha : G \longrightarrow L$, $\beta : H \longrightarrow L$ are homomorphisms such that $\alpha f = \beta g$, there exists a unique homomorphism $\gamma : P \longrightarrow L$ such that $p\gamma = \alpha$ and $q\gamma = \beta$.

To begin with consider the simplest example of this situation, namely where $K = \{1\}$ and the maps f and g are the obvious ones. In that case the requirements above are that whenever α and β are arbitrary homomorphisms into L there is a unique map from the group P into L, which factors both α and β. The group P in that case is easy to construct. Namely, consider the group whose elements are arbitrary words in alphabet from G and H and their inverses. Multiplication is given by juxtaposition and the only cancelation rules come from G and H. This group is called the free product of G and H and is denoted $G * H$. Before proceeding we discuss some examples.

Example 8.10. *Let $G = H = \mathbf{Z}/2\mathbf{Z}$ and let σ and τ denote the generators of G and H respectiely. Then the group $G * H$ is the collection of all words of the following forms 1, σ, τ ,$\sigma{*}\tau$, $\tau{*}\sigma$, $\sigma{*}\tau{*}\sigma$, $\tau{*}\sigma{*}\tau$, etc. Multiplication is given by juxtaposition, thus for instance $(\sigma * \tau) * (\sigma * \tau * \sigma) = \sigma * \tau * \sigma * \tau * \sigma$ and $(\sigma * \tau) * (\tau * \sigma * \tau) = \tau$.*

The free product of G and H has the feature that if K is an arbitrary group and there are homomorphisms $f : G \longrightarrow K$ and $g : H \longrightarrow K$ then there is a unique homomorphism from $G * H$ to K which agrees with f and g on the repective images of G and H in the free product. In some sense the free product of two groups is in the world of groups corresponds to the wedge operation in the world of pointed spaces and to disjoint union for unpointed spaces. We have sketched a proof in the tutorials of the fact that the fundamental group of a wedge of two pointed spaces is the free product of the respective fundamental groups. The Van Kampen theorem is just a generalisation of this obsetvation.

Definition 8.11. *Let G, H and K be groups and let homomorphisms $g : K \longrightarrow G$ and $h : K \longrightarrow H$ be given. The free amalgamated product of G and H over K is defined to be the quotient group of the free product $G * H$ by the minimal normal subgroup containing all elements of the form $g(k) * h(k)^{-1}$.*

Example 8.12. *The free product of G and H is the same as the free amalgamated product over the trivial group with g and h being the obvious homomorphisms.*

Many times free amalgamated products are easy to write in terms of generators and relations if a presentation for the groups involved is known.

Example 8.13. *Let $G = \mathbf{Z}/4\mathbf{Z}$ and $H = \mathbf{Z}/(\mathbf{Z}6)$, generated by x and y respectively. Let $K = \mathbf{Z}/2\mathbf{Z}$ generated by t and let $g(t) = x^2$ and $h(t) = y^3$ be the unique non-trivial homomorphisms from K into G and H repsetively. Then*

$$G *_K H = \mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z} = \langle x, y \mid x^4 = y^6 = 1, x^2 = y^3 \rangle.$$

Example 8.14. *Let G, H and K be as above but replace the homomorphism h by the trivial homomorphism. Then*

$$G *_K H = \langle x, y \mid x^4 = y^6 = 1, x^2 = 1 \rangle = \langle x, y \mid x^2 = y^6 = 1 \rangle = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/6\mathbf{Z}.$$

Definition 8.15. *Let G be a group. The abelianisation G_{ab} of G is the maximal abelian quotient group of G. It is obtained as the quotient group of G by the minimal normal subgroup containing all elements of the form $x^{-1}y^{-1}xy$ for $x, y \in G$.*

Notice that two isomorphic groups have isomorphic abelianisations. However, two groups with the same abelianisation may not be isomorphic. In particular the abelianisation of an abelian group G is G itself because every element of the form $x^{-1}y^{-1}xy$ is trivial. Thus $(G_{ab})_{ab} = G_{ab}$ but $G \neq G_{ab}$ unless G is abelian.

If G is given in terms of generators and relations then G_{ab} is presented by the same generators and for realtions one takes the original relations together with extra relations $x^{-1}y^{-1}xy = 1$, one for every pair of generators x and y. The reader might like to use the idea of abelianisation to convince himself that the two groups obtained above as free amalgamated products of $\mathbf{Z}/4\mathbf{Z}$ by $\mathbf{Z}/6\mathbf{Z}$ over $\mathbf{Z}/2\mathbf{Z}$ are not isomorphic. The abelianisation of the first is $\mathbf{Z}/12\mathbf{Z}$, whereas the second abelianises to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$.

Example 8.16. *Let F_1, F_2 be free groups on two generators x, y and u, v respectively. Let $K = \mathbf{Z}$ denote the group of integers (free group on one generator) generated by z. Let $f_i : K \longrightarrow F_i$ be defined by $f_1(z) = xyx^{-1}y^{-1}$ and $f_2(z) = uvu^{-1}v$. The free amalgamated product has a presentation*

$$F_1 *_K F_2 = \langle x, y, u, v \mid xyx^{-1}y^{-1} = uvu^{-1}v \rangle.$$

The free amalgamated product of G and H over K has the following property. Let F be an arbitrary group and let $\alpha : G \longrightarrow F$ and $\beta : H \longrightarrow F$ be homomorphisms such that $\alpha g = \beta h$. Let $i : G \longrightarrow G *_K H$ and $j : H \longrightarrow G *_K H$ denote the inclusions. Then there exists a unique homomorphism $\gamma : G *_K H \longrightarrow F$ such that $\gamma i = \alpha$ and $\gamma j = \beta$. We are now ready to state the Van Kampen Theorem.

Theorem 8.17. *Let $X = U \cup V$ with U, V and $U \cap V$ non-empty, open in X and path-connected. Assume the base point x_0 is in $U \cap V$. Let $u : U \longrightarrow X$ and $v : V \longrightarrow X$ denote the inclusions. Then the induced maps on fundamental groups induce an isomorphism*

$$\pi_1(X, x_0) \cong \pi_1(U, x_o) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0).$$

Before we prove the theorem we explain its statement and examine a few examples. The theorem states the following. Under the topological assumptions on U and V, consider the homomorphisms $g : \pi_1(U \cap V, x_0) \longrightarrow \pi_1(U, x_0)$ and $f : \pi_1(U \cap V, x_0) \longrightarrow \pi_1(V, x_0)$. With respect to those, one can construct the free amalgamated produc of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ over $\pi_1(U \cap V, x_0)$. Now, there are the maps $u_\# : \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0)$ and $v_\# : \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$ and by the mapping property of the free amalgamated product there exists a unique homomorphism from $\gamma : \pi_1(U, x_o) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$ to $\pi_1(X, x_0)$ such that $\gamma i = u_\#$ and $\gamma j = v_\#$. The theorem states that γ is an isomorphism.

Let us now consider a few examples.

Example 8.18. *Let X and Y be pointed spaces. Then the Van Kampen theorem gives an indirect proof that $\pi_1(X \vee Y, (x_0, y_0)) \cong \pi_1(X, x_0) * \pi_1(Y, y_0)$.*

The next family of examples come from connected sums of two closed surfaces. Recall that the connected sum construction is formed by removing an open disk from the respective surfaces and glueing them along the boundaries. Thus the connected sum of two surfaces can be though of as a union of two punctured surfaces intersecting on a circle. By thickening the circle of intersection a bit one can guarantee that the conditions of the theorem are satisfied. The next step is to compute the homomorphism induced by inclusion of the circle as the boundary of the respective punctured surface.

The first and easiest is the case where the surface is a 2-sphere. In that case the punctured surface is contractible and the induced map

$$\pi_1(S^1) \longrightarrow \pi_1(S^2 \setminus D^2)$$

is trivial.

Corollary 8.19. *The 2-sphere is simply-connected.*

Proof. The 2-sphere is obtained by attaching two 2-disks or equivalently a disk and a punctured 2-sphere along the boundary. By the Van Kampen theorem

$$\pi_1(S^2) \cong \pi_1(D^2) *_{\pi_1(S^1)} \pi_1(S^2 \setminus D^2) = \{1\}.$$

□

Next consider the torus T^2. We saw how the torus can be obtained from a square by identifying each pair of parallel edges ini the same orientation.

Lemma 8.20. *The punctured torus $T^2 \setminus D^2$ has the homotopy type of a wedge of two circles and the inclusion of the boundary $S^1 \longrightarrow T^2 \setminus D^2$ induces on fundamental groups the map which takes a generator of $\pi_1(S^1)$ to the element $a^{-1}b^{-1}ab$ where $\pi_1(T^2 \setminus D^2)$ is the free group on a and b.*

Proof. Consider the torus as the identification space of the square as above. Then the punctured torus can be thought of as the same identification space but with an open disk removed from the interior of the square. But this open disk can be continuously deformed to the entire interior of the square. Hence the punctured torus is homotopy equivalent to the boundary of the square with the appropriate identifications, which is easily seen to be a wedge of two circles.

Next consider the inclusion of the circle as the boundary of the punctured torus. Let z denote a generator of $\pi_1(S^1)$. We may choose a represetative for z to be the identity map ι on S^1. Let a denote the homotopy class of the loop α in the punctured torus given by tracing the left edge from top to bottom. Let b denote the class of the loop β given by tracing the bottom edge from right to left. Then the incusion j of the boundary is homotopic to the map which takes the circle to the loop $\alpha^{-1}\beta^{-1} * \alpha * \beta$. Thus $j_\#(z) = a^{-1}b^{-1}ab$ as claimed. □

Remark 8.21. *Notice that one may choose each one of the vertices of the square as a starting point (as a base point they are all the same) and also orient the identifications four different ways. This will result only in modifying the set of generators for $\pi_1(T^2 \setminus D^2)$.*

Next we present a calculation of the fundamental group of the torus using the Van Kampen theorem.

Corollary 8.22. $\pi_1(T^2) \cong \mathbf{Z} \times \mathbf{Z}$.

Proof. The torus can be thought of as obtained from the punctured torus by attaching a disk along the boundary. As before spaces can be "thikened" so that the conditions of the Van Kampen theorem are satisfied. Thus by the lemma one has

$$\pi_1(T^2) \cong (\mathbf{Z} * \mathbf{Z}) *_{\mathbf{Z}} \{1\} = \langle a, b \mid a^{-1}b^{-1}ab = 1\rangle \cong \mathbf{Z} \times \mathbf{Z}.$$

□

The Klein bottle K^2 can be obtained from a square by identifying one pair of parallel edges in the same orientation and the other in opposite orientation.

Lemma 8.23. *The punctured Klein bottle $K^2 \setminus D^2$ has the homotopy type of a wedge of two circles and the inclusion of the boundary $S^1 \longrightarrow K^2 \setminus D^2$ induces on fundamental groups the map which takes a generator of $\pi_1(S^1)$ to the element $a^{-1}b^{-1}ab^{-1}$ where $\pi_1(K^2 \setminus D^2)$ is the free group on a and b.*

Proof. the argument showing that the punctured Klein bottle is homotopy equivalent to a wedge of two circles is identical to the case of the punctured torus. However because of the different identifications on the boundary of the square one has $j_\#(z) = a^{-1}b^{-1}ab^{-1}$. □

Similar to the calculation of $\pi_1(T^2)$ we now present a calculation of $\pi_1(K^2)$ using the Van Kampen theorem.

Corollary 8.24. $\pi_1(K^2) \cong \langle a, b \mid a^{-1}b^{-1}ab^{-1} = 1\rangle$.

Proof. The argument is identical to the case of the torus, just the relation introduced by the Van Kampen theorem is different. □

Now at last we are able to show easily that the torus and the Klein bottle are not homeomorphic. In fact they are not even homotopy equivalent.

Corollary 8.25. *The torus and the Klein bottle have distinct homotopy type.*

Proof. One has $\pi_1(T^2) \cong \mathbf{Z}\times\mathbf{Z} \cong \langle a, b \mid a^{-1}b^{-1}ab = 1\rangle$ and $\pi_1(K^2) \cong \langle a, b \mid a^{-1}b^{-1}ab^{-1} = 1\rangle$. In particular $\pi_1(T^2)$ is abelian and $\pi_1(K^2)$ is not because in it $a^{-1}b^{-1}a = b$ and in an abelian group the corresponding equation will be $a^{-1}b^{-1}a = b^{-1}$. This together with the observation that $b \neq b^{-1}$ in $\pi_1(K^2)$ concludes the proof.

Another way to see the same is to consider the abelianisation of the fundamental groups under consideration. The group $\pi_1(T^2)$ is abelian so abelianisation doesn't change it. On the other hand if $\pi_1(K^2)$ is abelianised, then the relation $a^{-1}b^{-1}ab^{-1} = 1$ becomes $b^{-2} = 1$. Hence $(\pi_1(K^2))_{ab} \cong \mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, which is not isomorphic to $\mathbf{Z} \times \mathbf{Z}$. □

The real projective plane $\mathbf{R}P^2$ is the orbit space of the antipodal action on the 2-sphere.

Lemma 8.26. *The punctured real projective plane $\mathbf{R}P^2 \setminus D^2$ has the homotopy type of a circle and the inclusion of the boundary $S^1 \longrightarrow \mathbf{R}P^2 \setminus D^2$ induces on fundamental groups the map which takes a generator of $\pi_1(S^1)$ to the element a^2 where $\pi_1(\mathbf{R}P^2 \setminus D^2)$ generated by a.*

Proof. One may think of the real projective plane as the upper hemisphere with the boundary given by the equator subject to the antipodal identification. Notice that if one removes a small open disk from the interior of the hemisphere the resulting space is homotopy equivalent to the circle, given by the equator. This proves the first claim.

To see the second assertion of the lamma notice that the inclusion of the boundary in the punctured real projective plane is homotopic to the map which traces the equator exactly once. But the equator is subject to the given identification and so tracing it once is homotopic to tracing the generator of $pi_1(\mathbf{R}P^2 \setminus D^2)$ twice. The lemma follows. □

Corollary 8.27. $\pi_1(\mathbf{R}P^2) \cong \mathbf{Z}/2\mathbf{Z}$

Proof. This follows again by the Van Kampen theorem observing that

$$\pi_1(\mathbf{R}P^2) \cong \mathbf{Z} * \mathbf{Z}\{1\} \cong \langle a \mid a^2\rangle \cong \mathbf{Z}/2\mathbf{Z}.$$

□

Corollary 8.28. *The homotopy type of $\mathbf{R}P^2$ is distinct from those of T^2 and K^2.*

Proof. One has only to compare fundamental groups. The fundamental groups of T^2 and K^2 are infinite, whereas the fundamental group of $\mathbf{R}P^2$ is finite. □

We are now ready to glue surfaces together. This will be done in a rather random way. Namely, we will not try to be systematic in the way we form connected sums of surfaces and will use various combinations of the 2-sphere, torus, Klein bottle and the real projective plane. Later on we will consider the classification theorem for closed surfaces and will in particular be able to present a calculation using the Van Kampen theorem of all possible fundamental groups of closed surfaces. This will enable us to argue that all the surfaces described in the clasification theorem have distinct homotopy types.

Lemma 8.29.

$$\pi_1(T^2\#T^2) \cong \langle a, b, c, d \mid a^{-1}b^{-1}ab = c^{-1}d^{-1}cd\rangle.$$

Lemma 8.30.

$$\pi_1(T^2\#K^2) \cong \langle a, b, c, d \mid a^{-1}b^{-1}ab = c^{-1}d^{-1}cd^{-1}\rangle.$$

Lemma 8.31.

$$\pi_1(T^2\#\mathbf{R}P^2) \cong \langle a, b, c \mid a^{-1}b^{-1}ab = c^2\rangle.$$