UNIVERSITY OF ABERDEEN **SESSION 1998–99**

Degree Examination

MA1002 Calculus

January 1999 **(3pm—5pm)**

Attempt ALL FIVE of the questions in SECTION A and TWO of the questions in SECTION B. Each question in section A is worth 12 marks and each question in section B is worth 20 marks.

Calculators may be used ONLY for the arithmetic of real numbers or the numerical evaluation of trigonometric, logarithmic and exponential functions. Calculator memories must be clear at the start of the examination; in particular, the use of pre-stored programs is prohibited. Marks may be deducted for answers that do not show clearly how the solution is reached.

SECTION A

1. Differentiate the following functions:

$$f(x) = 4x^3 + 3x^{-2/3}, \qquad g(x) = \frac{1+x^2}{1+x}, \qquad h(x) = x^3\cos(x).$$

2. Evaluate the following integrals:

$$\int x^2 - x^{-3} + 7e^{4x}\,dx\,, \qquad \int_0^{\pi/2} \cos\left(\frac{\pi\sin x}{2}\right)\cos x\,dx\,.$$

Use integration by parts to show that

$$\int_0^1 \arctan x\,dx = \frac{\pi}{4} - \ln\sqrt{2}\,.$$

3. **(a)** Find the critical points of the function $f(x) = x^3 - 4x^2 + 4x$ and hence find the maximum and minimum values of this function on the range $0 \leqslant x \leqslant 1$.

(b) Show that the function $g(x) = \sin(x^2)$ has a critical point at x=0. Is it a maximum, a minimum or neither?

4. Use the Newton-Raphson method to find the solution to the equation

$$2x^3 + 3x^2 - 4x - 5 = 0$$

that is near $x = 1.3$. You should state your result accurate to 4 decimal places.

SECTION B

5. Show that the function $x = e^t \sin(2t)$ satisfies the equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 5x = 0$$

for all values of t.

6. **(a)** Say how many arbitrary constants you expect the general solution of a first order differential equation to contain, and give a brief reason for your answer.

Find a solution of the differential equation

$$\frac{dy}{dx} = 2x(1 + y^2)$$

for which $y = \pi/4$ when $x = 0$.

(b) Show that

$$\int_{-1}^{1} \frac{2x + 5}{(x + 2)(2x + 3)}\, dx = \ln\left(\frac{25}{3}\right).$$

7. **(a)** Differentiate the following functions.

$$a(x) = \ln\sqrt{1 + x^2}, \qquad b(x) = \arctan\left(\frac{1}{x}\right), \qquad c(x) = \exp(xe^x)$$

(b) Explain briefly why the following equations need qualification to make them true,

$$\arcsin(\sin(x)) = x, \qquad \sin(\arcsin(x)) = x.$$

8. A tank is in the form of a cylinder with a circular base. The volume of the tank is V. Find the values of the radius r and height h of the tank that minimise the overall area of the tank (curved sides and flat top and bottom).

The area of the curved surface of such a tank is $2\pi rh$.

UNIVERSITY OF ABERDEEN **SESSION 1998–99**

Degree Examination **SOLUTIONS**

MA1002 Calculus

January 1999 **(3pm—5pm)**

SECTION A

1. The derivatives are

$$f'(x) = 12x^2 - 2x^{-5/3}, \qquad g'(x) = \frac{x_{+}^{2}2x - 1}{(1+x)^2}$$

$$h'(x) = 3x^2\cos(x) - x^3\sin(x).$$

2. The first integral is

$$\frac{x^3}{3} + \frac{x^{-2}}{2} + \frac{7}{4}\mathrm{e}^{4x}.$$

In the second integral, let $u = \sin x$. Then $du = \cos x\, dx$, and

$$\int_0^{\pi/2} \cos\left(\frac{\pi \sin x}{2}\right) \cos x\, dx = \int_0^1 \cos\left(\frac{\pi u}{2}\right)\, du = \left[\frac{2}{\pi}\sin\left(\frac{\pi u}{2}\right)\right]_0^1 = \frac{2}{\pi}.$$

Finally, using integration by parts,

$$\int_0^1 \arctan x\, dx = \Big[x \arctan x\Big]_0^1 - \int_1^1 \frac{x}{1+x^2}\, dx.$$

Put $u = 1 + x^2$ in the second integral, so $du = 2x\, dx$. Then

$$\int_0^1 1.\arctan x\, dx = \frac{\pi}{4} - \int_1^2 \frac{du}{2u} = \frac{\pi}{4} - \frac{\ln(2)}{2} = \frac{\pi}{4} - \ln\sqrt{2}\,.$$

3. **(a)** The derivative is $f'(x) = 3x^2 - 8x + 4$ and this is zero when (solving the quadratic) $x = 2$ or $x = 2/3$. The only critical point in the specified range is $x = 2/3$. We have $f(0) = 0$, $f(1) = 1$ and $f(2/3) = 32/27$. So the maximum value is $32/27$ at $x = 2/3$ and the minimum is 0 at $x = 0$.

(b) $g'(x) = 2x\cos(x^2)$. This is certainly zero when $x = 0$. $g''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$. So $g''(0) = 2 > 0$ and we have a minimum. (Can also be done just as easily by looking at signs of g').

4. We consider the function $f(x) = 2x^3 + 3x^2 - 4x - 5$. We are solving $f(x) = 0$. The Newton-Raphson iteration step is then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Taking the hint, we start with $x_0 = 1.3$.

n	x_n	d_n
0	1.300000	0.052798
1	1.352798	0.002014
2	1.350784	0.000003
3	1.350781	7×10^{-10}

So the required solution would seem to be $x = 1.3508$ to 4 decimal places.

SECTION B

5. **(a)** Working out the derivatives we get

$$\begin{aligned} x &= e^t \sin(2t), \\ \dot{x} &= e^t(\sin(2t) + 2\cos(2t)) \\ \ddot{x} &= e^t(4\cos(2t) - 3\sin(2t)) \end{aligned}$$

So

$$\ddot{x} - 2\dot{x} + 5x = e^t(4\cos(2t) - 3\sin(2t) - 2\sin(2t) - 4\cos(2t) + 5\sin(2t)) = 0$$

as required.

6. **(a)** The general solution of a first order differential equation will usually contain a single arbitrary constant, because in general it will involve a single integration.

We can rewrite the given equation as

$$\frac{dy}{dx}\frac{1}{1+y^2} = 2x,$$

and so integrating both sides we have

$$\int \frac{dy}{1+y^2} = x^2 + K$$

for some constant of integration K. Thus

$$\arctan(y) = x^2 + K \quad \text{or equivalently} \quad y = \tan\left(x^2 + K\right).$$

When $x = 0$, $y = \tan(K)$, so if we take $K = 1$, we obtain a solution with the required property.

(b) Using partial fractions we write

$$\frac{2x+5}{(x+2)(2x+3)} = \frac{A}{x+2} + \frac{B}{2x+3}.$$

Thus $2x+5 = A(2x+3) + B(x+2)$ and this holds for all x. Putting $x = -2$, we have $1 = A(-1)$ and so $A = -1$. Equating the coefficient of x then shows that $2 = -2 + B$, so $B = 4$, and

$$\begin{aligned}\int_{-1}^{1} \frac{2x+5}{(x+2)(2x+3)}\,dx &= \int_{-1}^{1} \frac{4\,dx}{2x+3} - \int_{-1}^{1} \frac{dx}{x+2} \\ &= \Big[2\ln(2x+3) - \ln(x+2)\Big]_{-1}^{1} \\ &= 2\ln(5) - 2\ln(1) - \ln(3) + \ln(1) = \ln(25/3).\end{aligned}$$

7. **(a)** The derivatives are

$$a(x) = \ln\sqrt{1+x^2} = \frac{1}{2}\ln(1+x^2) \qquad \text{so} \qquad a'(x) = \frac{x}{1+x^2}$$

$b = \arctan(u)$ where $u = 1/x$. So

$$\frac{db}{du} = \frac{1}{1+u^2}, \qquad \frac{du}{dx} = -\frac{1}{x^2}$$

So

$$b'(x) = -\frac{1}{x^2(1+u^2)} = -\frac{1}{1+x^2}$$

Finally

$c = e^u$ where $u = xe^x$. so $c' = e^u$ and $u' = (1+x)e^x$, So, by chain rule, $c'(x) = (1+x)e^x \exp(xe^x)$.

(b) The first statement is only valid if x is within the conventional range of definition of arcsin. This is normally taken to be $[-\pi/2, \pi/2]$. Without this constraint the definition becomes ambiguous.

The second statement just needs the qualification that $-1 \leqslant x \leqslant 1$.

8. The volume V and area A are given in terms of the height h and the radius r by

$$V = \pi r^2 h \qquad \text{and} \qquad A = 2\pi r h + 2\pi r^2$$

We are given V, so we can write

$$h = \frac{V}{\pi r^2}$$

and therefore we can write A in terms of r as

$$A(r) = \frac{2V}{r} + 2\pi r^2$$

. We know that r is positive.

Differentiating, we get

$$A'(r) = -\frac{2V}{r^2} + 4\pi r$$

The derivative is only zero when $r^3 = V/2\pi$.

For values of r below this the derivative is clearly negative, and becomes positive afterwards. So we have a global minimum at

$$r = \left(\frac{V}{2\pi}\right)^{1/3}$$

and, putting this back into the earlier expression,

$$h = 2\left(\frac{V}{2\pi}\right)^{1/3} = 2r.$$