Author	igc
Date	6.2.03
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UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICAL SCIENCES

MX3503 LINEAR OPTIMISATION AND NUMERICAL ANALYSIS
SHEET 3: EXPLORING BASIC SOLUTIONS

1. The following tableau displays the coefficients of six columns $\mathbf{a}_1, \dots, \mathbf{a}_6$, relative to a basis $\mathbf{a}_4$, $\mathbf{a}_2$, $\mathbf{a}_5$. Insert first $\mathbf{a}_1$ for $\mathbf{a}_5$ and then $\mathbf{a}_3$ for $\mathbf{a}_4$, and so obtain a tableau displaying the coefficients of $\mathbf{a}_1, \dots, \mathbf{a}_6$, relative to the basis $\mathbf{a}_3$, $\mathbf{a}_2$, $\mathbf{a}_1$. Why cannot the replacement of $\mathbf{a}_4$ by $\mathbf{a}_3$ be made first?

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
$\mathbf{a}_4$	2	0	0	1	0	2
$\mathbf{a}_2$	5	1	3	0	0	0
$\mathbf{a}_5$	-1	0	4	0	1	6

2. For the system

$$\begin{aligned} 6x_1 - x_2 + 14x_3 - 20x_4 + 7x_5 &= -16, \\ 2x_1 + x_2 - 5x_3 + 10x_4 - 3x_5 &= 11, \\ x_1 + x_2 - 7x_3 + 12x_4 - 4x_5 &= 13, \end{aligned}$$

use the tableau format to find first the basic solution in which x_1 and x_4 are zero. Then, by making a change of basis, find the basic solution in which x_1 and x_5 are zero.

3. *This questions is designed to help you with manipulation in the general case. It considers a special case of a change of basic columns where we assume that* $\mathbf{a}_1$ *is known to be the column that is leaving the basis.*

Suppose that $\mathbf{A}$ is an $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, where $m < n$ and the rank of $\mathbf{A}$ is equal to m. Suppose that $\mathbf{a}_1, \dots, \mathbf{a}_m$ form a basis for the column space of $\mathbf{A}$ and let y_{ij} be the coefficient of $\mathbf{a}_i$ in the expression of $\mathbf{a}_j$ as a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_m$. In order that $\mathbf{a}_r$ $(r > m)$ can replace $\mathbf{a}_1$ in the basis, what must be true? Assuming this is true, express the column $\mathbf{a}_1$ in terms of $\mathbf{a}_r, \mathbf{a}_2, \dots, \mathbf{a}_m$.

Hence obtain the expression

$$\mathbf{a}_j = \frac{y_{1j}}{y_{1r}}\mathbf{a}_r + \sum_{i=2}^{m} \left(y_{ij} - \frac{y_{ir}}{y_{1r}} y_{1j} \right) \mathbf{a}_i,$$

for $\mathbf{a}_j$ in terms of $\mathbf{a}_r, \mathbf{a}_2, \dots, \mathbf{a}_m$. Verify that this agrees with what is obtained by applying suitable row operations to the appropriate tableau.

4. The tableau displays a basic feasible solution of a linear programming problem in standard form.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$
$\mathbf{a}_2$	1	1	0	4	0	-7	2
$\mathbf{a}_5$	3	0	0	9	1	6	6
$\mathbf{a}_3$	-2	0	1	5	0	2	3

(a) Obtain a new basic feasible solution by inserting the vector $\mathbf{a}_4$ into the basis.

(b) Obtain a new basic feasible solution by inserting $\mathbf{a}_6$ (instead of $\mathbf{a}_4$) into the original basis.

(c) Obtain a new basic feasible solution by inserting $\mathbf{a}_1$ into the original basis.

5. In (c) of the previous question, a degenerate basic feasible solution was obtained, but not in (a) or (b). Prove the following generalisation of what this has illustrated. Suppose that a linear programming problem in standard form and a non-degenerate basic feasible solution are given, and a vector is to be inserted into the basis to obtain a new basic feasible solution. Then the new solution is degenerate if and only if there is a tie for smallest among the ratios u_i/y_{is} considered to determine which basis vector leaves.

6. The tableau displays a **degenerate** basic feasible solution of some linear programming problem.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{a}_7$	$\mathbf{a}_8$	$\mathbf{a}_9$	$\mathbf{b}$
$\mathbf{a}_4$	0	-2	0	1	4	0	1	0	-3	2
$\mathbf{a}_1$	1	8	0	0	0	0	-2	0	-4	0
$\mathbf{a}_3$	0	6	1	0	2	0	3	0	2	3
$\mathbf{a}_6$	0	5	0	0	3	1	1	0	4	4
$\mathbf{a}_8$	0	1	0	0	-1	0	-4	1	0	0

Which of the basic variables are zero? Which of the four non-basic vectors can be inserted into the basis to obtain a **non-degenerate** basic feasible solution? Carry out the replacement and obtain the non-degenerate basic feasible solution.

7. Suppose that, given a **degenerate** basic feasible solution of a linear programming problem in standard form (as in the previous question), a vector is chosen to enter the basis and the vector to leave is determined as in the simplex method. Find necessary and sufficient conditions for the resulting basic feasible solution to be **non-degenerate**, with a proof to substantiate your answer.

University of Aberdeen
Department of Mathematical Sciences

Author	igc
Date	6.2.03
Code	mth201-102984-6

MX3503 Linear Optimisation and Numerical Analysis
Solutions to Sheet 3: Exploring Basic Solutions

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1. The resulting tableau is given in Table 1.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
$\mathbf{a}_3$	0	0	1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{7}{4}$
$\mathbf{a}_2$	0	1	0	$-\frac{23}{8}$	$-\frac{3}{4}$	$-\frac{41}{4}$
$\mathbf{a}_1$	1	0	0	$\frac{1}{2}$	0	1

Table 1: Final tableau for Example 1.

The replacement of $\mathbf{a}_4$ by $\mathbf{a}_3$ cannot be made first because $y_{13} = 0$.

2. Our initial system can be written, in tableau form, as

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{b}$
	6	-1	14	-20	7	-16
	2	1	-5	10	-3	11
	1	1	-7	12	-4	13

We pivot in order to get a basis of the column space consisting of $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_5$. It is convenient to introduce a_2 to the basis first, to get the tableau:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{b}$
	7	0	7	-8	3	-3
	1	0	2	-2	1	-2
$\mathbf{a}_2$	1	1	-7	12	-4	13

.

Next introduce $\mathbf{a}_5$:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{b}$
	4	0	1	-2	0	3
$\mathbf{a}_5$	1	0	2	-2	1	-2
$\mathbf{a}_2$	5	1	1	4	0	5

.

Finally, we have no choice — introduce $\mathbf{a}_3$ to the basis.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{b}$
$\mathbf{a}_3$	4	0	1	-2	0	3
$\mathbf{a}_5$	-7	0	0	2	1	-8
$\mathbf{a}_2$	1	1	0	6	0	2

.

This gives solution $(0, 2, 3, 0, -8)$. To find the solution with $x_1 = x_5 = 0$, we swap $\mathbf{a}_4$ into the basis and remove $\mathbf{a}_5$:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{b}$
$\mathbf{a}_3$	-3	0	1	0	1	-5
$\mathbf{a}_4$	-7/2	0	0	1	1/2	-4
$\mathbf{a}_2$	22	1	0	0	-3	26

.

This gives solution $(0, 26, -5, -4, 0)$.

3. We use Theorem 2.15 on page 18 of the notes to see that we must have $y_{1r} \neq 0$; that the coefficient of $\mathbf{a}_r$ when expressed in terms of the basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\ldots$, $\mathbf{a}_m$ of the column space, should be non-zero.

For any $j > m$, we have

$$\mathbf{a}_j = y_{1j}\mathbf{a}_1 + \sum_{i=2}^{m} y_{i,j}\mathbf{a}_i. \tag{1}$$

If in addition, we have $y_{1r} \neq 0$, we can write

$$\mathbf{a}_1 = \frac{1}{y_{1r}}\mathbf{a}_r - \sum_{i=2}^{m} \frac{y_{ir}}{y_{1r}}\mathbf{a}_i.$$

Substituting this value for $\mathbf{a}_1$ in equation 1, we have

$$\mathbf{a}_j = \frac{y_{1j}}{y_{1r}}\mathbf{a}_r + \sum_{i=2}^{m} \left(y_{ij} - \frac{y_{ir}}{y_{1r}} y_{1j} \right) \mathbf{a}_i$$

as required.

4. **(a)** Considering ratios of coefficients of $\mathbf{a}_4$ and $\mathbf{b}$ gives 2/4, 6/9 and 3/5. We thus replace $\mathbf{a}_2$ by $\mathbf{a}_4$:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	a_6	$\mathbf{b}$
$\mathbf{a}_4$	1/4	1/4	0	1	0	-7/4	1/2
$\mathbf{a}_5$	3/4	-9/4	0	0	1	87/4	3/2
$\mathbf{a}_3$	-13/4	-5/4	1	0	0	43/4	1/2

.

This gives solution $(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 0)$.

(b) Again computing ratios, we swap $\mathbf{a}_6$ in to replace $\mathbf{a}_5$:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	a_6	$\mathbf{b}$
$\mathbf{a}_2$	9/2	1	0	29/2	7/6	0	9
$\mathbf{a}_6$	1/2	0	0	3/2	1/6	1	1
$\mathbf{a}_3$	-3	0	1	2	-1/3	0	1

.

This gives solution $(0, 9, 1, 0, 0, 1)$.

(c) Here, computing ratios shows there is a choice; we can replace $\mathbf{a}_2$ or $\mathbf{a}_5$. We choose to replace $\mathbf{a}_2$:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	a_6	$\mathbf{b}$
$\mathbf{a}_1$	1	1	0	4	0	-7	2
$\mathbf{a}_5$	0	-3	0	-3	1	27	0
$\mathbf{a}_3$	0	2	1	13	0	-12	7

.

This gives solution $(2, 0, 7, 0, 0, 0)$.

5. Consider the situation in Table 2.3 on page 20 of the notes, where we consider the situation that occurs on moving from one basic solution to a new one. Assuming that the original solution was non-degenerate, we may suppose that $u_i >$ for all i. If we move to a degenerate basic solution, then for some i, we will have

$$u_i - y_{is}(u_r/y_{rs}) = 0$$

and so, since $u_i \neq 0$, it follows that $y_{is} \neq 0$, and hence that $(u_i/y_{is}) = (u_r/y_{rs})$ as claimed.

Conversely, if there *is* such a tie, between $\mathbf{a}_i$ and $\mathbf{a}_r$ say, then if we insert $\mathbf{a}_r$ into the basis, then x_i moves from u_i to $u_i - y_{is}(u_r/y_{rs}) = 0$, and we move to a *degenerate* basic solution.

6. In the given tableau x_1 and x_8 are basic variables which are both zero. To get a non-degenerate basic feasible solution, there is only one possible replacement, namely $\mathbf{a}_7$, giving $(2, 0, 0, 1, 0, 3, 1, 4, 0)$.

To see why this is the only solution, start with Theorem 2.16 of the notes. This result is about replacing one element by another in a basis. Since we wish a distinct basic feasible solution (it has to be different from the one we have if it is to be non-degenerate), we can only pivot about those $y_{rs} > 0$ for which

$$\frac{u_r}{y_{rs}} = \min_i \left\{ \frac{u_i}{y_{is}} : y_{is} > 0 \right\} \neq 0.$$

Calculating the minimum on the right hand side above, we may not insert $\mathbf{a_2}$ because the minimum is zero. Inserting $\mathbf{a_5}$, the minimum test makes us pivot about 4, to replace $\mathbf{a_4}$; this means that x_1 remains zero after the substitution, and so we do not move to a non-degenerate solution. In the same way, inserting $\mathbf{a_9}$, the minimum test makes us pivot about 4, to replace $\mathbf{a_6}$; this means that x_8 remains zero after the substitution, and so we do not move to a non-degenerate solution. The remaining possibility is that we insert $\mathbf{a_7}$; the minimum test means we pivot about 3 to replace $\mathbf{a_3}$, and get the basic feasible solution shown.

7. Suppose that $\mathbf{a}_s$ is chosen to enter the basis. The resulting basic feasible solution is non-degenerate if and only if $y_{is} < 0$, for each i for which $u_i = 0$, and there is no tie for smallest among the ratios u_i/y_{is}.