UNIVERSITY OF ABERDEEN **SESSION 1998–99**

Degree Examination

MX3503 Linear Optimization and Numerical Analysis

Thursday 27 May 1999 **(3pm to 5pm)**

Attempt THREE questions

Calculators may be used ONLY for the arithmetic of real numbers or the numerical evaluation of trigonometric, logarithmic and exponential functions. Calculator memories must be clear at the start of the examination; in particular, the use of pre-stored programs is prohibited. Marks may be deducted for answers that do not show clearly how the solution is reached.

1. **(a)** A pharmaceutical company is creating a tablet for a new drug. Each tablet is to contain a binder, a disintegrant and a filler in addition to the active drug ingredient, which is to be 14% of the weight of each tablet. Chemical and physical considerations mean that the weight of the disintegrant should not exceed 25% of the combined weights of the binder and the active ingredient, and that there should be at most 10 times as much filler as binder. The disintegrant costs £15, the binder £50 and filler £2 per kilogram.

The problem is to decide how to formulate the tablet in order to minimise its cost. Express the problem as a linear programming problem. [You are **not** asked to solve the problem.]

(b) Use the Simplex Algorithm to maximise $-3x_1 + 4x_2$, subject to $x_1, x_2 \geqslant 0$ and

$$\begin{aligned} 2x_1 - x_2 &\geqslant -5, \\ x_1 + 3x_2 &\leqslant 22. \end{aligned}$$

Either give the maximum value of the objective function, and values of x_1 and x_2 at which this maximum is obtained, or explain briefly why there is no maximum.

2. Explain how the simplex method, when applied to an unbounded linear programming problem, will eventually reveal the fact that it is unbounded.

Use the Simplex Algorithm to minimise $-6x_1 + 2x_2 - 3x_3$, subject to $x_1, x_2, x_3 \geqslant 0$ and

$$\begin{aligned} x_1 + x_2 - 2x_3 &\leqslant 2, \\ -4x_1 + x_2 - x_3 &\geqslant 6, \\ x_1 + 3x_2 - 8x_3 &\leqslant 8. \end{aligned}$$

Describe briefly the reasoning behind each step.

3. Suppose that two matrix games are given by

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 8 & 6 \\ 2 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 6 \\ 8 & 4 \\ 4 & 2 \end{pmatrix}$$

where the matrices give the payoff to Rowman. Obtain the upper and lower values α and β for each game, and say which game is strictly determined.

Use the simplex method to solve the other game, showing that the value of the game lies between α and β. Give the optimum strategy for each of the two players.

If you were Rowman, would you, given the choice, play game **A** or game **B**? Why?

4. The function $z = x_1 - x_2$ is to be minimised subject to the constraints that

$$\begin{aligned} x_1^2 + x_2^2 &\leqslant 1, \\ 2x_1^2 - 3x_2 &\leqslant 0. \end{aligned}$$

Sketch the feasible region. Hence obtain geometrically the optimal value and the corresponding values of x_1 and x_2.

Write down the Kuhn-Tucker conditions for this problem. Illustrate their use by deriving the solution you have just obtained. What information do you get about the tightness of the constraints?

UNIVERSITY OF ABERDEEN **SESSION 1998–99**

Degree Examination SOLUTIONS

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1. **(a)** Let b, d and f be the number of kilograms of binder, disintegrant and filler in each 100 kilograms of the formulation. Then since there will be 14 kilograms of active ingredient in each 100 kilograms of the formulation, $b + d + f = 86$. The binder - filler constraint is that $10b \geqslant f$, while the constraint on the disintegrant gives $4d \leqslant b + 14$. These, together with the reality requirement, that $b \geqslant 0$, $d \geqslant 0$ and $f \geqslant 0$ are all the constraints, and the problem is to minimise the total cost $C = 50b + 15d + 2f$ subject to these constraints.

(b) We first introduce slack variables and convert to tableau form.

```
> with(linalg):
> A:=matrix(3,2,[-2,1,1,3,3,-4]):
> B:=concat(A,diag(1,1,1),vector([5,22,0]));
```

$$B := \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 5 \\ 1 & 3 & 0 & 1 & 0 & 22 \\ 3 & -4 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Since this is a maximising problem, there is only one proper sign for improvement, in column 2. Looking at the appropriate ratios shows we should swap $\mathbf{a}_3$ out from the basis. Pivoting, we have:-

```
> C:=pivot(B,1,2);
```

$$C := \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 5 \\ 7 & 0 & -3 & 1 & 0 & 7 \\ -5 & 0 & 4 & 0 & 1 & 20 \end{bmatrix}$$

Again there is only one proper sign for improvement. This time there is no choice of ratios, since one of the two relevant entries is negative and so ignored.

```
> E:=mulrow(C,2,1/7);
```

$$E := \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 5 \\ 1 & 0 & \dfrac{-3}{7} & \dfrac{1}{7} & 0 & 1 \\ -5 & 0 & 4 & 0 & 1 & 20 \end{bmatrix}$$

```
> F:=pivot(E,2,1);
```

$$F := \begin{bmatrix} 0 & 1 & \frac{1}{7} & \frac{2}{7} & 0 & 7 \\ 1 & 0 & \frac{-3}{7} & \frac{1}{7} & 0 & 1 \\ 0 & 0 & \frac{13}{7} & \frac{5}{7} & 1 & 25 \end{bmatrix}$$

There are now no proper signs for improvement, showing the algorithm has converged. the maximum value is 25 and this is attained when $x_1 = 1$ and $x_2 = 7$.

2. If the problem is unbounded, eventually a column will occur in which the row corresponding to the objective function will have the proper sign for improvement, but everything above that column will be an inappropriate pivot, in that each entry is negative or zero. Even if there are appropriate pivots in other columns, this, by itself, shows that the problem is unbounded in that suitable multiples of the solution corresponding to that column can be added to the existing solution to make it as large (in the required sense) as chosen.

To solve the given problem we first introduce slack variables. In doing this we see that the second constraint generates a bad row; we move it down so it comes after all the good rows.

```
> with(linalg):
> A:=matrix(4,3,[1,1,-2,1,3,-8,4,-1,1,6,-2,3]):
> B:=concat(A,diag(1,1,1,1),vector([2,8,-6,0]));
```

$$B := \begin{bmatrix} 1 & 1 & -2 & 1 & 0 & 0 & 0 & 2 \\ 1 & 3 & -8 & 0 & 1 & 0 & 0 & 8 \\ 4 & -1 & 1 & 0 & 0 & 1 & 0 & -6 \\ 6 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Treating the bad row as the objective function, there is only one proper sign for improvement, in column 2; the corresponding ratios are 2 and 8/3 so we swap $\mathbf{a}_4$ out of the basis.

```
> C:=pivot(B,1,2);
```

$$C := \begin{bmatrix} 1 & 1 & -2 & 1 & 0 & 0 & 0 & 2 \\ -2 & 0 & -2 & -3 & 1 & 0 & 0 & 2 \\ 5 & 0 & -1 & 1 & 0 & 1 & 0 & -4 \\ 8 & 0 & -1 & 2 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This still hasn't made the bad row good. This time, there is a proper sign for improvement in column 3, and no choice of pivot row above it. In this situation we know we can make the bad row good by pivoting in the bad row itself, which we now do.

```
> E:=mulrow(C,3,-1):E=pivot(E,3,3);
```

$$E = \begin{bmatrix} -9 & 1 & 0 & -1 & 0 & -2 & 0 & 10 \\ -12 & 0 & 0 & -5 & 1 & -2 & 0 & 10 \\ -5 & 0 & 1 & -1 & 0 & -1 & 0 & 4 \\ 3 & 0 & 0 & 1 & 0 & -1 & 1 & 8 \end{bmatrix}$$

Now we see the situation we discussed at the start of the question. We are minimising the objective function, which thus has a proper sign for improvement in column 1. All the entries above it are negative, so we can make the objective function as large and negative as we wish.

3. Let $\mathbf{A} = [a_{ij}]$ and recall that if $\alpha = \max_i \min_j a_{ij}$ and $\beta = \min_j \max_i a_{ij}$, then α and β are respectively the **lower** and **upper** values for the matrix game $\mathbf{A}$.

In our case,

$$\begin{array}{ccc} & & \text{Min} \\ \mathbf{A} = & \begin{pmatrix} 4 & 2 \\ 8 & 6 \\ 2 & 4 \end{pmatrix} & \begin{array}{c} 2 \\ 6 \\ 2 \end{array} \\ \text{Max} & \begin{array}{cc} 8 & 6 \end{array} & \end{array} \quad \text{while} \quad \begin{array}{ccc} & & \text{Min} \\ \mathbf{B} = & \begin{pmatrix} 2 & 6 \\ 8 & 4 \\ 4 & 2 \end{pmatrix} & \begin{array}{c} 2 \\ 4 \\ 2 \end{array} \\ \text{Max} & \begin{array}{cc} 8 & 6 \end{array} & \end{array}$$

and so for $\mathbf{A}$, we have $\alpha = \beta = 6$ and the game is strictly determined, while for $\mathbf{B}$ we have $\alpha = 4$, while $\beta = 6$.

We solve the game $\mathbf{B}$ using the simplex method, so we seek a $\mathbf{y}$ which is an optimum strategy for Columnman; ie we maximise $y_1 + y_2$ subject to $\mathbf{By} \leqslant \mathbf{1}$ and $y_1 \geqslant 0$, $y_2 \geqslant 0$.

The initial tableau is

```
> B:=matrix(4,2,[2,6,8,4,4,2,-1,-1]):
> B:=concat(B,diag(1,1,1,1),vector([1,1,1,0]));
```

$$B := \begin{bmatrix} 2 & 6 & 1 & 0 & 0 & 0 & 1 \\ 8 & 4 & 0 & 1 & 0 & 0 & 1 \\ 4 & 2 & 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Following the simplex algorithm, we choose to pivot about the element 8 in column 1, since each of columns 1 and 2 have the proper sign for improvement. Pivoting gives the tableau

```
> C:=pivot(B,2,1):C:=mulrow(C,2,1/8);
```

$$C := \begin{bmatrix} 0 & 5 & 1 & \frac{-1}{4} & 0 & 0 & \frac{3}{4} \\ 1 & \frac{1}{2} & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & \frac{-1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & \frac{-1}{2} & 0 & \frac{1}{8} & 0 & 1 & \frac{1}{8} \end{bmatrix}$$

Continuing, we see that column 2 has the proper sign for improvement, and we must pivot about 5, giving the tableau

```
> E:=pivot(C,1,2):E:=mulrow(E,1,1/5);
```

$$E := \begin{bmatrix} 0 & 1 & \frac{1}{5} & \frac{-1}{20} & 0 & 0 & \frac{3}{20} \\ 1 & 0 & \frac{-1}{10} & \frac{3}{20} & 0 & 0 & \frac{1}{20} \\ 0 & 0 & 0 & \frac{-1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{10} & \frac{1}{10} & 0 & 1 & \frac{1}{5} \end{bmatrix}$$

The optimum strategy is thus $\frac{5}{1}\left(\frac{1}{20}, \frac{3}{20}\right) = \left(\frac{1}{4}, \frac{3}{4}\right)$ for Columnman. The optimal strategy for Rowman is $\frac{5}{1}\left(\frac{1}{10}, \frac{1}{10}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

The expected payoff if both players use this strategy is thus

$$\frac{1}{4}\left(\frac{2}{2} + \frac{8}{2}\right) + \frac{3}{4}\left(\frac{6}{2} + \frac{4}{2}\right) = 5.$$

The first game thus has a larger expected payoff to Rowman, and so is the logical choice.

4. We show below the two constraints and two different positions of the objective function

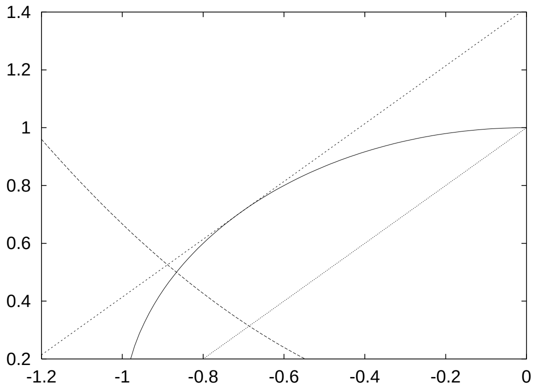

Figure 1: The feasible region is above the parabola and below the circle.

The objective function becomes smaller as the intercept on the x_2 - axis increase; in other words, as the line moves to the northwest!

The Lagrangian is

$$L(x_1, x_2, \lambda, \mu) = x_1 - x_2 - \lambda(1 - x_1^2 - x_2^2) - \mu(3x_2 - 2x_1^2).$$

Using these equations, The Kuhn-Tucker conditions are thus $\lambda \geqslant 0$, $\mu \geqslant 0$ and

$$\begin{aligned} 1 + \lambda 2x_1 + 4\mu x_1 &= 0, & -1 + 2\lambda x_2 - 3\mu &= 0, \\ 1 - x_1^2 - x_2^2 &\geqslant 0, & 2x_1^2 - 3x_2 &\geqslant 0, \\ \lambda(1 - x_1^2 - x_2^2) &= 0, & \mu(2x_1^2 - 3x_2) &= 0. \end{aligned}$$

and solutions to these equations give the local minimisers — strictly unless a certain tangent condition is non-degenerate, as it is in this case.

The first equation shows that $\lambda = \mu = 0$ does not give a solution; thus at least one of the constraints is tight.

If $\lambda = 0$, the second equation gives $\mu = -1/3$, which contradicts the fact that $\mu \geqslant 0$.

If $\lambda > 0$ and $\mu > 0$ so both constraints are tight, x_1 and x_2 satisfy

$$x_1^2 + x_2^2 = 1, \qquad 2x_1 = 3x_2.$$

Eliminating x_1 gives

$$3x_2 + 2x_2^2 - 2 = 0 = (2x_2 - 1)(x_2 + 2)$$

and $x_2 = 1/2$, since the other solution, $x_2 = -2$ does not give a real solution for x_1. This gives $x_1 = \pm\sqrt{3}/2$ and a minimum value for z of $-(\sqrt{3} - 1)/2$.

Finally we consider the case $\mu = 0$, $\lambda > 0$ so $x_1^2 + x_2^2 = 1$. From our diagram we expect this to give the global minimum. Since $\mu = 0$, the first equation gives $1 + 2\lambda x_1 = 0$, and $x_1 = -1/(2\lambda)$, while the second equation gives $x_2 = 1/(2\lambda)$. Since $\lambda > 0$, the "circle" constraint, that $x_1^2 + x_2^2 \leqslant 1$ is tight (ie an equality) and $2\lambda^2 = 1$. This corresponds to the solution $x_1 = -1/\sqrt{2}$, $x_2 = 1/\sqrt{2}$ and an objective value of $z = -\sqrt{2}$ which is thus the minimum.

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Degree Examination SETTER'S COMMENTS

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The paper is starting to be influenced by the availability of MAPLE. *I have tried to avoid dirty calculation, while testing that principles are understood. And I am starting to push the ability to translate a problem from words, rather than simply execute an algorithm.*

1. Problem setup and the one-phase algorithm.

(a) This is almost straightforward, except for having the amount of active ingredient fixed, and is a new question of a type that has been seen and set before. I think it important that they work on formulation; this question is very easy, and is designed to encourage this work. There is potential difficulty in interpreting the filler - binder constraint. **[8 marks]**

(b) This one-phase problem is almost exactly the same as the one set for CA (the actual question is new). It tests the whole of the algorithm, and they have to make choices. Note that neither choice is easier than the other. This should prove a banker. **[12 marks]**

2. Two-phase problems; bad rows, artificial variables etc.

(a) The bookwork is really there to remind them of how to do the main part of the question. There is no proof involved, so should be straightforward. **[4 marks]**

(b) A tableau with a bad row which doesn't become good after a single pivot! They haven't to my knowledge seen such a question before, although they have met all the components. This is very much a test of "putting it all together". **[16 marks]**

3. Duality and game theory

Nothing special. The numbers are probably about as easy as they can be. I want to emphasise the whole process; although I guess some will get the numbers wrong.

Upper and lower values etc **[6 marks]**

The main problem. **[12 marks]**

Which would you play? **[2 marks]**

4. This problem is original. They have had no practice doing non-linear optimisation geometrically, although they have done linear examples, and this objective function is linear. I hope the question emphasise **understanding** the Kuhn-Tucker conditions.

Graphical solution. **[6 marks]**

General solution. **[12 marks]**

Tightness. **[2 marks]**