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MA2004 Sets and Algebraic Structures
Solutions to Examples 2

1. There are $n(n-1)(n-2)\ldots(n-m+1)$ injective functions. This can be rewritten as $n!/(n-m)!$.

2. (i) The primes are a subset of $\mathbb{N}$ and hence countable.

(ii) The complex numbers contain the real numbers, and are hence uncountable by Cantor's theorem.

(iii) There is an injective function from ordered pairs of natural numbers to natural numbers given by $(m,n) \mapsto 2^m.3^n$, so they are countable.

(iv) We can show that the functions from $\mathbb{N}$ to $\mathbb{N}$ are uncountable as follows. Suppose that they are countable, and that $f_0, f_1, \ldots$ is a list of the functions. We form a new function $f\colon \mathbb{N} \to \mathbb{N}$ by decreeing that $f(n) = 0$ if $f_n(n) \neq 0$ and $f(n) = 1$ if $f_n(n) = 0$. Then f is different from all the listed functions.

(v) We can show that the subsets of $\mathbb{N}$ are uncountable as follows. Suppose that we they are countable, and that $X_0, X_1, X_2, \ldots$ is a list of the subsets. We form a new subset X by decreeing that $0 \in X$ if and only if $0 \notin X_0$, $1 \in X$ if and only if $1 \notin X_1$, and so on. Then X is different from all the listed subsets.

3. Since $87 < 100 = 10^2$, we need to check for divisibility by 2,3,5 and 7. In fact, 87 is divisible by 3, so it is not prime.

Since $103 < 121 = 11^2$, we need to check for divisibility by 2,3,5 and 7. So 103 is a prime.

Since $169 = 13^2$, it is not a prime.

4. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

5. If the sum of two primes is odd, then one has to be even and the other has to be odd. The only even prime is 2, so the only possibility is $999 = 2 + 997$. We check that 997 is indeed a prime by checking that it isn't divisible by 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 or 31.

6. $30000 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 5 \times 5 = 2^4.3.5^4$.

7. $\gcd(26, 42) = 2$, $s = -8$ and $t = 5$.

8. To pay 2 pence, you give the shopkeeper five 42 pence tokens and the shopkeeper gives you back eight 26 pence tokens. To pay 6 pence, you pay fifteen 42 pence tokens and get back twenty-four 26 pence tokens. You can't pay for an item costing 5 pence.

9. The greatest common divisor is 11. Here's the calculation copied from my calculator (which happens to be the `bc` command in `unix`, but it could just as easily have been a hand-held calculator):

```
a=11290345
b=192874
a/b
58.53741302612067982205
```

```
c=a-58*b
c
103653
d=b-c
d
89221
e=c-d
e
14432
d/e
6.18216463414634146341
f=d-6*e
f
2629
e/f
5.48953974895397489539
g=e-5*f
g
1287
h=f-2*g
h
55
g/h
23.40000000000000000000
i=g-23*h
i
22
j=h-2*i
j
11
quit
```

10. (i) n is the square of an integer if and only if all the a_i are even.

(ii) If $n = (u/v)^2 = u^2/v^2$ with n, u and v integers, then $nv^2 = u^2$. From part (i), every prime appearing in the factorisation of n occurs with even multiplicity, so n is the square of an integer.

11. Eight o'clock.

12.

+	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	]3]	[4]	[5]	[0]
[2]	[2]	]3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

×	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

One example is $a = 3$, $b = 1$ and $c = 3$. There are plenty of other examples.