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MA2004 Sets and Algebraic Structures
Solutions to Examples 1

1. (i) An element of $(X \cap Y) \cup Z$ is in both $X \cup Z$ and $Y \cup Z$, so it is in $(X \cup Z) \cap (Y \cup Z)$. Conversely, an element of $(X \cup Z) \cap (Y \cup Z)$ is either in Z or it is in both X and Y, so it is in $(X \cap Y) \cup Z$.

(ii) An element of $(X \cup Y) \cap Z$ is either in X and in Z or it is in Y and in Z, so it is in $(X \cap Z) \cup (Y \cap Z)$. Conversely, an element of $(X \cap Z) \cup (Y \cap Z)$ is in $X \cup Y$ and it is in Z so it is in $(X \cup Y) \cap Z$.

2. $(\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\} \ \cup \ \{\varnothing, \{\varnothing, \{\varnothing\}\}\}) \ \cap \ \{\{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}\}$
$= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \ \cap \ \{\{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}\}$
$= \{\{\varnothing\}, \{\{\varnothing\}\}\}$

3. $X \times Y$ has mn elements.

4. (i) There are $2^3 = 8$ functions.

(ii) There are n^m functions.

5. (i) There are 24 bijective functions.

(ii) If $m = n$ then there are $n!$ bijective functions. Otherwise there are none.

6. (i) If $f(x_1) = f(x_2)$ then $g(f(x_1)) = g(f(x_2))$. So if f is not injective then $g \circ f$ is not injective.

(ii) Given $z \in Z$, if $g \circ f$ is surjective then there exists $x \in X$ such that $g(f(x)) = z$. So $f(x)$ is an element of Y whose image under g is z, which shows that g is surjective.

7. Define a function $f\colon Z \to X \times Y$ by $f(z) = (f_1(z), f_2(z))$. Then this has the required properties. Furthermore, if f is any function satisfying the required properties, then the first coordinate of $f(z)$ has to be $f_1(z)$ and the second has to be $f_2(z)$.

If W has the properties given in the second half of the question, then taking W for Z in the first half of the question gives us a function $g\colon W \to X \times Y$ satisfying $\pi_1 \circ g = \phi_1$ and $\pi_2 \circ g = \phi_2$. On the other hand, taking $X \times Y$ as the Z for the defining property of W, we obtain a function $h\colon X \times Y \to W$ satisfying $h \circ \phi_1 = \pi_1$ and $h \circ \phi_2 = \pi_2$. Now $g \circ h$ is a function from $X \times Y$ to $X \times Y$ such that $g \circ h \circ \pi_1 = \pi_1$ and $g \circ h \circ \pi_2 = \pi_2$. Since the identity function 1 on $X \times Y$ also satisfies $1 \circ \pi_1 = \pi_1$ and $1 \circ \pi_2 = \pi_2$, uniqueness shows that $g \circ h = 1$. In a similar fashion we get $h \circ g = 1$, the identity function on W. This implies that g and h are inverse functions, so each is a bijection.